A construction of rigid analytic cohomology classes for congruence subgroups of $\mathrm{SL}_3(\mathbb{Z})$

David Pollack and Robert Pollack

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1 Introduction

Let $k \geq 0$ be an integer and let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. By Eichler-Shimura theory, the cohomology group $H^1(\Gamma, \mathrm{Sym}^k(\mathbb{C}))$, considered as a Hecke-module, contains the space of cuspforms of level Γ and weight $k+2$. In [12], Stevens studied a much larger cohomology group, one with coefficients in a space of p-adic distributions $\mathbf{D}_k$ equipped with a weight k action of the Iwahori subgroup $\Gamma_0(p) \subseteq \mathrm{SL}_2(\mathbb{Z})$. The space $\mathbf{D}_k$ admits an equivariant map to $\mathrm{Sym}^k(\mathbb{Q}_p^2)$ and thus yields a Hecke-equivariant map

$$H^1(\Gamma_0, \mathbf{D}_k) \xrightarrow{\rho_k} H^1(\Gamma_0, \mathrm{Sym}^k(\mathbb{Q}_p^2))$$

on cohomology. Here $\Gamma_0 = \Gamma_0(p) \cap \Gamma$. While the target of ρ_k encodes information about classical modular forms, the source contains information about overconvergent modular forms of weight $k+2$ (see [13]). Moreover, in [12], Stevens proved that if one restricts to the subspace where U_p acts with non-critical slope, the above map becomes an isomorphism. (This should be viewed as the analogue of Coleman's theorem on small slope overconvergent forms being classical.)

Ash and Stevens in [5, 6] generalized the above comparison theorem to representations of $\mathrm{GL}_n(\mathbb{Q})$ (even to arbitrary $\mathbb{Q}_p$-split reductive groups). If λ is a character of a torus of GL_n, let V_λ denote the $\mathbb{Q}_p$-representation with highest weight λ. One then replaces $\mathbf{D}_k$ with a space of p-adic distributions $\mathbf{D}_\lambda$ endowed with a weight λ action. As before, this space maps equivariantly to V_λ and Ash and Stevens proved that the corresponding map

$$H^r(\Gamma_0, \mathbf{D}_\lambda) \xrightarrow{\rho_\lambda} H^r(\Gamma_0, V_\lambda)$$

is an isomorphism if one restricts to a subspace where the slope of U_p is small enough.

A more explicit study of the case of $\mathrm{GL}_2(\mathbb{Q})$ was made in [10] via modular symbols. As a consequence of the above comparison theorem, any non-critical classical Hecke-eigensymbol lifts to a unique $\mathbf{D}_k$-valued Hecke-eigensymbol. In

[10], the following constructive proof of this fact is given: form an arbitrary lift of the classical eigensymbol to a $\mathbf{D}_k$-valued modular symbol (but not necessarily to an eigensymbol). Explicit formulae for such lifts are given. Then iterate the U_p-operator to obtain a sequence that converges to the sought after $\mathbf{D}_k$-valued Hecke-eigensymbol.

One may hope then to use the methods of [10] to explicitly lift classical Hecke-eigenclasses for $\mathrm{GL}_n(\mathbb{Q})$ with $n > 2$. One daunting part of such a task is generalizing the first step of lifting a V_λ-valued cohomology class to a $\mathbf{D}_\lambda$-valued cohomology class. For $\mathrm{GL}_2(\mathbb{Q})$, this was done by explicitly "solving the Manin relations" which required writing down an explicit fundamental domain for the action of a congruence subgroup on the upper-half plane. To repeat these arguments for $\mathrm{GL}_n(\mathbb{Q})$ with $n > 2$ would involve examining the geometry of certain higher dimensional symmetric spaces which on the surface appears to be a difficult task.

However, in the case of $\mathrm{GL}_2(\mathbb{Q})$, M. Greenberg [9] simplified the arguments of [10] and managed to form liftings in a "geometry-free" manner. His basic idea is to lift modular symbols into a larger ambient space. This larger space is big enough that forming such lifts is trivial. He then uses the U_p-operator to force such lifts back into the space of interest. Iterating this process leads to a sequence of modular symbols that converges to the true Hecke-eigensymbol. These ideas were used by Trifković in [14] to compute liftings of eigenclasses corresponding to automorphic forms for $\mathrm{GL}_2(K)$ with $K/\mathbb{Q}$ an imaginary quadratic field; this again is a situation where lifting classes directly is made difficult by the complicated geometry that is present.

In this paper, we generalize these constructions to the cohomology of $\mathrm{GL}_3(\mathbb{Q})$. As a rich theory of p-adic automorphic forms for higher rank groups is beginning to emerge, we note that there are very few groups simple enough for which computations and numerical exploration are feasible. Along with $\mathrm{Sp}_4(\mathbb{Q})$ and $\mathrm{GL}_2(K)$ for $K/\mathbb{Q}$ imaginary quadratic, $\mathrm{GL}_3(\mathbb{Q})$ is a natural next step in complexity beyond $\mathrm{GL}_2(\mathbb{Q})$. It is a complicated enough group so that many of the new higher rank phenomena are observable in its theory, but well enough understood that computational techniques exist for studying its V_λ-valued cohomology.

To carry out M. Greenberg's lifting idea in the context of GL_3 cohomology, we first axiomatize the situation as follows. Let R be a commutative ring and let $\Gamma \subseteq G$ be groups. Let $\pi \in G$ be such that Γ and $\pi^{-1}\Gamma\pi$ are commensurable. Set S to be the semigroup generated by Γ and π. Then the double coset $\Gamma\pi\Gamma$ induces an operator U on $H^r(\Gamma, M)$ for any right $R[S]$-module M. Consider a surjective map $D \to V$ of $R[S]$-modules and the induced map on cohomology

$$H^r(\Gamma, D) \to H^r(\Gamma, V).$$

Let φ be a U-eigenclass in $H^r(\Gamma, V)$ whose U-eigenvalue is a unit in $R/\operatorname{Ann}_R(\varphi)$. We prove the existence of a unique U-eigenclass Φ in $H^r(\Gamma, D)$ lifting φ under the assumption that D has a decreasing $R[S]$-stable filtration $\{F^n D\}$ such that:

1. $D/F^0D = V$,

2. $F^n D \cdot \pi \subseteq F^{n+1} D$,

3. The topology on D induced by $\{F^n D\}$ is separated and complete.

(This result corresponds to Theorem 3.1 in the paper.)

For $\mathrm{GL}_2(\mathbb{Q})$, a filtration on $\mathbf{D}_k$ that satisfies the above conditions is given in [10] and it is precisely these properties that are used in [9] to produce explicit lifts of modular symbols. In this paper, we construct for every dominant weight λ of $\mathrm{GL}_3(\mathbb{Q})$ a filtration on $\mathbf{D}_\lambda$ that satisfies the above axioms. In particular, we obtain another proof of the theorem of Ash and Stevens on lifting ordinary eigenclasses.

The proof of this general lifting theorem follows the methods of [9]. Indeed, the basic idea is to lift a V-valued cocycle representing φ to a D-valued cochain. There is no reason why such a lift should again be a cocycle. However, as $V = D/F^0 D$, this cochain will be a cocycle mod $F^0 D$. Applying the U-operator and dividing by the U-eigenvalue of φ forms a new cochain that still lifts φ, but is now a cocycle modulo $F^1 D$. This increase in accuracy is a consequence of the second assumption on the filtration. Iterating this process then leads to a sequence of cochains that converges to a cocycle whose image in cohomology is the desired Hecke-eigenlift.

For $\Gamma \subseteq \mathrm{SL}_3(\mathbb{Z})$, work of Ash and others [1, 4] gives a description of $H^3(\Gamma, M)$ in terms of GL_3-modular symbols. These spaces are computable using the methods of [4, 3], and by iterating the U_p-operator one can actually compute approximations to $\mathbf{D}_\lambda$-valued lifts of Hecke-eigenclasses in $H^3(\Gamma, V_\lambda)$. We carried out such computations for some boundary classes of small level and trivial weight V_0, and indeed obtained sequences of improving approximations to $\mathbf{D}_0$-valued Hecke eigenlifts of these classes. We intend to compute lifts of weight V_0 classes not arising from GL_2 found in [4] in the near future.

Now that the beginnings of a computational theory exists for $\mathrm{GL}_3(\mathbb{Q})$ many questions arise. For $\mathrm{GL}_2(\mathbb{Q})$, a non-critical Hecke-eigensymbol in $H^1(\Gamma_0, \mathbf{D}_k)$ encodes the p-adic L-function of the corresponding classical cuspform. Do these $\mathbf{D}_\lambda$-valued Hecke-eigenclasses encode some kind of p-adic L-function of the corresponding automorphic form? In [11], $\mathbf{D}_k$-valued Hecke-eigensymbols are used to attach a p-adic L-function to a critical slope modular form. Can critical slope $\mathrm{GL}_3(\mathbb{Q})$-forms be studied using these methods? In [8], the algorithms of [10] were used to compute Stark-Heegner points on elliptic curves. Can one hope to use these Hecke-eigenlifts to (conjecturally) construct points on the (conjectural) motive attached to these automorphic forms á la Darmon [7]?

The format of the paper is as follows: in the following section we introduce the distribution spaces $\mathbf{D}_\lambda$ for $\mathrm{GL}_3(\mathbb{Q})$. In the third section, we prove our general lifting result. In the fourth section, we construct a filtration on $\mathbf{D}_\lambda$ satisfying the properties mentioned above and obtain a lifting result for $\mathrm{GL}_3(\mathbb{Q})$.

2 p-adic distributions

In this section, we recall the notion of Ash and Stevens of p-adic distribution valued cohomology for $\mathrm{GL}_3(\mathbb{Q})$. Our description will be fairly concrete, proving many of the basic facts by explicit computations. We refer the reader to [5] and the forthcoming [6] for the case of $\mathrm{GL}_n(\mathbb{Q})$ and for a more systematic treatment.

2.1 Notation

Let p be a positive prime. Let $\mathbb{C}_p$ denote the completion of a fixed algebraic closure of $\mathbb{Q}_p$ and let $\mathbb{O}_p$ denote its ring of integers.

Let G denote the algebraic group scheme GL_3. Let B (resp. B^{opp}) denote the group of upper (resp. lower) triangular matrices in G. Let N (resp. N^{opp}) denote the group of unipotent matrices in B (resp. B^{opp}). Let T be the group of diagonal matrices so that $B = TN$ and $B^{\mathrm{opp}} = N^{\mathrm{opp}}T$

Let I denote the Iwahori subgroup of $G(\mathbb{O}_p)$, that is, the collection of elements in $G(\mathbb{O}_p)$ whose reduction modulo the maximal ideal of $\mathbb{O}_p$ is upper triangular.

Let $\Gamma_0(p)$ denote the Iwahori subgroup of $\mathrm{SL}_3(\mathbb{Z})$. If Γ is any congruence subgroup of $\mathrm{SL}_3(\mathbb{Z})$, we write Γ_0 for the intersection $\Gamma_0(p) \cap \Gamma$.

2.2 Spaces of distributions

Let X denote the image of I in $N^{\mathrm{opp}}(\mathbb{C}_p)\backslash G(\mathbb{C}_p)$. Since $I = (I \cap N^{\mathrm{opp}}(\mathbb{O}_p))B(\mathbb{O}_p)$, we have that X is isomorphic to $B(\mathbb{O}_p)$.

Let λ denote some algebraic character of the torus T. Consider the collection of $\mathbb{O}_p$-valued functions

$$M_\lambda := \{ f : X \to \mathbb{O}_p \ \mid \ f(tg) = \lambda(t) f(g) \text{ for } t \in T(\mathbb{O}_p) \text{ and } g \in X \}.$$

We wish to consider the subset of these functions that are $\mathbb{Q}_p$-rigid analytic. To make this precise, note that $N(\mathbb{O}_p)$ maps injectively into X. We give $N(\mathbb{O}_p)$ the structure of a $\mathbb{Q}_p$-rigid analytic space by identifying it with the unit polydisc in $\mathbb{C}_p^3$ via

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \in X \longleftrightarrow (x, y, z) \in \mathbb{O}_p^3.$$

So explicitly, a function on $N(\mathbb{O}_p)$ is $\mathbb{Q}_p$-rigid analytic if it is of the form

$$f \left(\begin{smallmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{smallmatrix} \right) = \sum_{i,j,k} c_{ijk} x^i y^j z^k$$

where $c_{ijk} \to 0$ as $i + j + k \to \infty$.

We then define

$$\mathbf{A}_\lambda := \left\{ f : X \to \mathbb{O}_p \ \middle| \ \begin{array}{c} f \text{ restricted to } N(\mathbb{O}_p) \text{ is a } \mathbb{Q}_p\text{-rigid function,} \\ f(tg) = \lambda(t) f(g) \text{ for } t \in T(\mathbb{O}_p) \end{array} \right\}.$$

Note that any function in $\mathbf{A}_\lambda$ is uniquely determined by its restriction to $N(\mathbb{O}_p)$.

Under our identification of $N(\mathbb{O}_p)$ with $\mathbb{O}_p^3$, rigid functions on $N(\mathbb{O}_p)$ correspond to elements of the Tate algebra $\mathbb{Z}_p\langle X, Y, Z\rangle$. Let f_{abc} denote the unique extension to $\mathbf{A}_\lambda$ of the function that sends $\left(\begin{smallmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{smallmatrix}\right)$ to $x^a y^b z^c$. Under the above identification, f_{abc} corresponds to the element $X^a Y^b Z^c$. Since the $\mathbb{Z}_p$-span of these monomials is dense in $\mathbb{Z}_p\langle X, Y, Z\rangle$, the span of the f_{abc} forms a dense subset of $\mathbf{A}_\lambda$.

We then set $\mathbf{D}_\lambda = \mathrm{Hom}_{cont}(\mathbf{A}_\lambda, \mathbb{Z}_p)$, the space of continuous $\mathbb{Z}_p$-linear functionals of $\mathbf{A}_\lambda$ into $\mathbb{Z}_p$. By the above observations, an element $\mu \in \mathbf{D}_\lambda$ is uniquely determined by its values on f_{abc} for all $a, b, c \geq 0$.

2.3 The weight λ action

Let S' be any semigroup of $M_3(\mathbb{Z}) \cap \mathrm{GL}_3(\mathbb{Z}_p)$ containing $\Gamma_0(p)$ and such that every element of S' is upper-triangular modulo p. Let π be the diagonal matrix with diagonal entries 1, p and p^2 and let S to be the semigroup generated by S' and π.

Note that S' acts on $N^{\mathrm{opp}}(\mathbb{O}_p)\backslash G(\mathbb{O}_p)$ by multiplication on the right which is easily seem to induce an action on X. We extend this to an action of S by letting π act by $N^{\mathrm{opp}} g \cdot \pi := N^{\mathrm{opp}} \pi^{-1} g \pi$. This is well-defined as π normalizes N^{opp} and again induces an action on X.

We then get a left action of S on M_λ by $(\gamma f)(g) = f(g \cdot \gamma)$. The following lemma describes this action explicitly on the functions f_{abc}. In particular, it will imply that this action induces a left action on $\mathbf{A}_\lambda$ and thus a right action on $\mathbf{D}_\lambda$ by $(\mu|\gamma)(f) = \mu(\gamma f)$.

Let $\lambda(k_1, k_2, k_3)$ be the character of the torus that sends $\begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$ to $d_1^{k_1} d_2^{k_2} d_3^{k_3}$. Also, for $f \in M_\lambda$, we write $f\left(\begin{smallmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{smallmatrix}\right) = f(x, y, z)$.

Lemma 2.1. *Let $\lambda = \lambda(k_1, k_2, k_3)$. For $\gamma \in S'$, the weight λ action of γ on $f \in M_\lambda$ is given by:*

$$
\begin{aligned}
&(\gamma f)(x, y, z) = \\
&\quad \det(\gamma)^{k_3}(a_{11} + a_{21}x + a_{31}y)^{k_1 - k_2}(m_{33} - m_{13}y - m_{23}z + m_{13}xz)^{k_2 - k_3} \\
&\quad f\left(\frac{a_{12} + a_{22}x + a_{32}y}{a_{11} + a_{21}x + a_{31}y}, \frac{a_{13} + a_{23}x + a_{33}y}{a_{11} + a_{21}x + a_{31}y}, \frac{-m_{32} + m_{12}y + m_{22}z - m_{12}xz}{m_{33} - m_{13}y - m_{23}z + m_{13}xz}\right)
\end{aligned}
$$

Here m_{ij} is the ij-th minor of γ. Also

$$(\pi f)(x, y, z) = f(px, p^2 y, pz).$$

Remark 2.2. In the case of $\mathrm{GL}_2(\mathbb{Q})$, if λ corresponds to $\mathrm{Sym}^k(\mathbb{Q}_p^2)$, then the weight λ action is simply given by

$$(\gamma f)\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) = (\gamma f)(x) = (a + cx)^k f\left(\frac{b + dx}{a + cx}\right).$$

Proof of Lemma 2.1. For $\gamma \in S'$, we have

$$\begin{aligned}
&\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} + a_{21}x + a_{31}y & a_{12} + a_{22}x + a_{32}y & a_{13} + a_{23}x + a_{33}y \\ a_{21} + a_{31}z & a_{22} + a_{32}z & a_{23} + a_{33}z \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\
&= \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \equiv \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & \frac{b_{22}b_{11}-b_{21}b_{12}}{b_{11}} & \frac{b_{23}b_{11}-b_{21}b_{13}}{b_{11}} \\ 0 & \frac{b_{32}b_{11}-b_{31}b_{12}}{b_{11}} & \frac{b_{33}b_{11}-b_{31}b_{13}}{b_{11}} \end{pmatrix} \\
&\equiv \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & \frac{b_{22}b_{11}-b_{21}b_{12}}{b_{11}} & \frac{b_{23}b_{11}-b_{21}b_{13}}{b_{11}} \\ 0 & 0 & \frac{\det(\gamma)}{b_{22}b_{11}-b_{21}b_{12}} \end{pmatrix} \\
&\equiv \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & \frac{b_{22}b_{11}-b_{21}b_{12}}{b_{11}} & 0 \\ 0 & 0 & \frac{\det(\gamma)}{b_{22}b_{11}-b_{21}b_{12}} \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{b_{12}}{b_{11}} & \frac{b_{13}}{b_{11}} \\ 0 & 1 & \frac{b_{23}b_{11}-b_{21}b_{13}}{b_{22}b_{11}-b_{21}b_{12}} \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

The congruences above are taking place in X – i.e. modulo $N^{\mathrm{opp}}(\mathbb{C}_p)$ (on the left).

Thus,

$$\begin{aligned}
&(\gamma f) \left(\begin{smallmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{smallmatrix}\right) \\
&= \lambda \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & \frac{b_{22}b_{11}-b_{21}b_{12}}{b_{11}} & 0 \\ 0 & 0 & \frac{\det(\gamma)}{b_{22}b_{11}-b_{21}b_{12}} \end{pmatrix} f \begin{pmatrix} 1 & \frac{b_{12}}{b_{11}} & \frac{b_{13}}{b_{11}} \\ 0 & 1 & \frac{b_{23}b_{11}-b_{21}b_{13}}{b_{22}b_{11}-b_{21}b_{12}} \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

A direct computation finds that

$$b_{23}b_{11} - b_{21}b_{31} = -m_{32} + m_{12}y + m_{22}z - m_{12}xz$$

and

$$b_{22}b_{11} - b_{21}b_{12} = m_{33} - m_{13}y - m_{23}z + m_{13}xz$$

where m_{ij} are the minors of γ. Plugging back in establishes the first formula of the lemma.

For π, we have

$$(\pi f) \left(\begin{smallmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{smallmatrix}\right) = f \left(\pi^{-1} \left(\begin{smallmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{smallmatrix}\right) \pi\right) = f \left(\begin{smallmatrix} 1 & px & p^2y \\ 0 & 1 & pz \\ 0 & 0 & 1 \end{smallmatrix}\right)$$

as claimed. □

Corollary 2.3. *The action of S on M_λ preserves $\mathbf{A}_\lambda$.*

Proof. In the formulae of Lemma 2.1, the only possibly troubling terms are $(a_{11} + a_{21}x + a_{31}y)^{-1}$ and $(m_{33} - m_{13}y - m_{23}z + m_{13}xz)^{-1}$. For γ in $S' \subset I$, one checks that a_{11} and m_{33} are units while all other coefficients present are divisible by p. In particular, the power series expansion of these two functions is again rigid analytic in x, y and z. $\square$

2.4 Specialization to weight λ

This section follows closely [5, Section 4] where the general case of GL_n is treated.

Let λ be an algebraic character of the torus T which is dominant with respect to the Borel B^{opp} and let V_λ be the finite dimensional representation of G with highest weight λ (with respect to B^{opp}). Fix $v_\lambda \in V_\lambda(\mathbb{Q}_p)$ a highest weight vector; that is, $v_\lambda \cdot t = \lambda(t) v_\lambda$ for $t \in T(\mathbb{Q}_p)$ and $v_\lambda \cdot n = v_\lambda$ for $n \in N^{\mathrm{opp}}(\mathbb{Q}_p)$.

Thus, the function

$$f_\lambda : G(\mathbb{O}_p) \to V_\lambda(\mathbb{C}_p) \text{ given by } f_\lambda(g) = v_\lambda \cdot g$$

descends to a function on $N^{\mathrm{opp}}(\mathbb{O}_p)\backslash G(\mathbb{O}_p)$ and restricts to give a function on X.

Lemma 2.4. *The function f_λ restricted to X is in $\mathbf{A}_\lambda \otimes V_\lambda(\mathbb{Q}_p)$.*

Proof. For $t \in T(\mathbb{O}_p)$ and $g \in G(\mathbb{O}_p)$,

$$f_\lambda(tg) = v_\lambda \cdot tg = \lambda(t) v_\lambda \cdot g = \lambda(t) f_\lambda(g)$$

since v_λ is a weight vector of weight λ. From this, it follows that f_λ is in $M_\lambda \otimes V_\lambda(\mathbb{Q}_p)$.

Further, let $v_1, v_2, \ldots, v_d$ be a basis of $V_\lambda(\mathbb{Q}_p)$. Then since V_λ is an algebraic representation of GL_3,

$$v_\lambda \cdot \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \sum_i P_i(x, y, z) v_i$$

where $P_i(x, y, z) \in \mathbb{Q}_p[x, y, z]$. Since these coefficients are polynomials, they are in particular $\mathbb{Q}_p$-rigid analytic. Hence, f_λ is in $\mathbf{A}_\lambda \otimes V_\lambda$. $\square$

We evaluate a distribution $\mu \in \mathbf{D}_\lambda$ on functions in $\mathbf{A}_\lambda \otimes V_\lambda(\mathbb{Q}_p)$ by setting $\mu(\sum f_i \otimes v_i)$ to be $\sum \mu(f_i) v_i$ where $\{v_i\}$ is a basis of V. It is clear that this definition is independent of the choice of basis.

Evaluation at f_λ then gives a map $\mathbf{D}_\lambda \to V_\lambda(\mathbb{Q}_p)$ which we will see below is S'-equivariant. However, this map is not π-equivariant and for this reason we introduce the $\star$-action on V_λ as in [2].

For $v \in V_\lambda$, we define

$$v \star \gamma = v \cdot \gamma \text{ for } \gamma \in S' \quad \text{and} \quad v \star \pi = \lambda(\pi)^{-1} v \cdot \pi.$$

This action extends uniquely to an action of S and we write $V_\lambda^\star$ when we view V_λ as a S-module under this action.

We now have to following analogue of Lemma 4.1 of [5].

Lemma 2.5. *Evaluating at f_λ gives an S-equivariant map*

$$\rho_\lambda : \mathbf{D}_\lambda \to V_\lambda^\star(\mathbb{Q}_p).$$

Proof. Note that for any $\gamma \in S'$,

$$f_\lambda(x\gamma) = v_\lambda \cdot (x\gamma) = (v_\lambda \cdot x) \cdot \gamma = f_\lambda(x) \cdot \gamma.$$

Recall that π acts on X by $x \cdot \pi = \pi^{-1} x \pi$. Thus,

$$f_\lambda(x \cdot \pi) = f_\lambda(\pi^{-1} x \pi) = v_\lambda \cdot (\pi^{-1} x \pi) = \lambda(\pi^{-1}) f_\lambda(x) \cdot \pi = f_\lambda(x) \star \pi.$$

We have thus proven that for all $\gamma \in S$

$$\gamma f_\lambda = e_\gamma \circ f_\lambda$$

where $e_\gamma : V_\lambda \to V_\lambda$ is the linear map $e_\gamma(v) = v \star \gamma$.

Now, if $L : V_\lambda(\mathbb{Q}_p) \to V_\lambda(\mathbb{Q}_p)$ is a linear map, one sees immediately that

$$\mu(L \circ f) = L(\mu(f))$$

for f in $\mathbf{A}_\lambda \otimes V_\lambda(\mathbb{Q}_p)$. Thus,

$$(\mu\big|\gamma)(f_\lambda) = \mu(\gamma f_\lambda) = \mu(e_\gamma \circ f_\lambda) = e_\gamma(\mu(f_\lambda)) = \mu(f_\lambda) \star \gamma$$

which proves the S-equivariance of ρ_λ. □

Set L_λ equal to the image of $\mathbf{D}_\lambda$ under this map; then L_λ is an S-stable $\mathbb{Z}_p$-lattice of $V_\lambda^\star(\mathbb{Q}_p)$. Let $\Gamma \subseteq \mathrm{SL}_3(\mathbb{Z})$ be a congruence subgroup and let $\Gamma_0 := \Gamma \cap \Gamma_0(p)$. The map ρ_λ then induces a map on cohomology

$$\rho_\lambda^r : H^r(\Gamma_0, \mathbf{D}_\lambda) \to H^r(\Gamma_0, L_\lambda)$$

which we refer to as specialization to weight λ.

These cohomology groups carry a natural action of Hecke operators $T(\ell, k)$ for ℓ a prime and $k = 1, 2, 3$. (See, for instance, [2]). We will primarily be interested in the operator U_p associated to the diagonal matrix π whose diagonal entries are 1, p and p^2. This operator will be carefully defined in the following section. We point out that since ρ_λ is S-equivariant, ρ_λ^r is automatically a Hecke-equivariant map.

The following theorem of Ash and Stevens analyzes the specialization map restricted to the U-ordinary subspace. (See [6] for a statement of the result in the form given here and also for the non-ordinary case; see [5] and [2] for analogous results involving Γ cohomology.)

Theorem 2.6. *The natural map*

$$\rho^r_\lambda : H^r(\Gamma_0, \mathbf{D}_\lambda)^{\mathrm{ord}} \xrightarrow{\sim} H^r(\Gamma_0, L_\lambda)^{\mathrm{ord}}.$$

is an isomorphism. Here M^{ord} denotes the direct sum of all generalized U_p-eigenspaces whose U_p-eigenvalue is a p-adic unit.

As a consequence of this theorem, any U-ordinary Hecke-eigensymbol in $H^r(\Gamma_0, L_\lambda)$ lifts uniquely to a Hecke-eigensymbol in $H^r(\Gamma_0, \mathbf{D}_\lambda)$. In the following sections, we will prove this fact in a constructive manner analogous to M. Greenberg's work in [9].

3 Lifting cohomology classes

In this section, we present a general lemma on lifting cohomology classes. The notation of this section is meant to mirror that of the previous section with the aim of making transparent the intended application to the case of interest.

Let $\Gamma \subseteq G$ be two groups, let π be some element in G, and let S be the sub-semigroup of G generated by Γ and π. Let R be a commutative ring and let D be any right $R[S]$-module. For $\gamma \in S$ and $\mu \in d$, we write the action of S on D by $\mu \cdot \gamma$.

If we assume that Γ and $\pi^{-1}\Gamma\pi$ are commensurable, there is an operator $U = U(\pi)$ on $H^i(\Gamma, D)$ defined as follows. Let D^π denote the $\pi^{-1}\Gamma\pi$-module whose underlying set is just D and whose action by $s \in \pi^{-1}\Gamma\pi$ is given by $\mu \cdot_\pi s = \mu \cdot \pi s \pi^{-1}$. Acting by π gives a map

$$H^r(\Gamma, D) \xrightarrow{\pi} H^r(\pi^{-1}\Gamma\pi, D^\pi)$$

and restriction and transfer yield maps

$$H^r(\pi^{-1}\Gamma\pi, D^\pi) \xrightarrow{res} H^r(\Delta, D^\pi) \xrightarrow{tr} H^r(\Gamma, D)$$

where $\Delta = \Gamma \cap \pi^{-1}\Gamma\pi$. We define U as the composition of these three maps.

For x in an R-module M, we set $\mathrm{Ann}_R(x)$ to be the ideal of elements of R that annihilate x. If M is a right R-module and $T : M \to M$ is a linear map, we call a non-zero element $x \in M$ an eigenvector for T with eigenvalue $\alpha \in R$, if $x \big| T = \alpha x$. Note that α is only determined modulo $\mathrm{Ann}_R(x)$. We say that x is ordinary for T, if the image of α is a unit in $R/\mathrm{Ann}_R(x)$.

The following theorem is the main result of the section.

Theorem 3.1. *Let D be a right $R[S]$-module with a decreasing $R[S]$-filtration $\{F^n D\}$ such that*

1. *$F^n D \cdot \pi \subseteq F^{n+1} D$ for each $n \geq 0$,*
2. *the natural map $D \to \varprojlim D/F^n D$ is an isomorphism.*

Let φ be in $H^r(\Gamma, D/F^0 D)$ be an ordinary eigenvector for U with eigenvalue α. Then there exists $\Phi \in H^r(\Gamma, D)$ such that

1. the image of Φ *in* $H^r(\Gamma, D/F^0D)$ *equals* φ,

2. Φ *is an eigenvector for* U *with eigenvalue* α,

3. $\mathrm{Ann}_R(\Phi) = \mathrm{Ann}_R(\varphi)$.

Moreover, if Φ' *is any ordinary* U*-eigenlift of* φ*, then* $\Phi' = \Phi$.

The proof of this theorem will occupy the remainder of the section. We will make use of a non-canonical lift of U to the level of cochains. To this end, let

$$\cdots \to F_r \xrightarrow{d_r} F_{r-1} \to \cdots \to F_0 \to R \to 0$$

be a free resolution of R by right $R[\Gamma]$-modules. Applying $\mathrm{Hom}_\Gamma(-, D)$ yields

$$0 \to \mathrm{Hom}_\Gamma(F_0, D) \to \cdots \to \mathrm{Hom}_\Gamma(F_{r-1}, D) \xrightarrow{d_r} \mathrm{Hom}_\Gamma(F_r, D) \to \dots .$$

Set

$$C^r(\Gamma, D) = \mathrm{Hom}_\Gamma(F_r, D), \; Z^r(\Gamma, D) = \ker(d_{r+1}), \text{ and } B^r(\Gamma, D) = \mathrm{im}(d_r).$$

So, by definition, $H^r(\Gamma, D) \cong Z^r(\Gamma, D)/B^r(\Gamma, D)$.

Note that $F_\bullet^\pi \to R \to 0$ is a free resolution of $R[\pi^{-1}\Gamma\pi]$-modules. Also, both $F_\bullet \to R \to 0$ and $F_\bullet^\pi \to R \to 0$ are free $R[\Delta]$-resolutions of R. In particular, there exists an $R[\Delta]$-chain complex map τ:

$$\begin{array}{ccccc} F_\bullet & \longrightarrow & R & \longrightarrow & 0 \\ \downarrow{\scriptstyle \tau_\bullet} & & \downarrow{\scriptstyle =} & & \\ F_\bullet^\pi & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

lifting the identity map on R.

Unraveling the definition of restriction and transfer gives the following description of U on the level of cocycles, as in [5, Formulae 4.3]. Let $\varphi \in H^r(\Gamma, D)$ and let $\tilde{\varphi} \in Z^r(\Gamma, D)$ be a cocycle representing φ. Decompose the double coset $\Gamma\pi\Gamma$ as a union of right cosets $\cup_i \Gamma\pi\gamma_i$ for $\gamma_i \in \Gamma$. Then

$$(\tilde{\varphi}|U)(f_r) \equiv \sum_i \tilde{\varphi}(\tau_r(f_r \cdot \gamma_i^{-1})) \cdot \pi\gamma_i \pmod{B^r(\Gamma, D)}.$$

To lift U to the level of cochains, we simply define an operator $U : \mathrm{Hom}(F_r, D) \to \mathrm{Hom}(F_r, D)$ by

$$(\varphi|U)(f_r) := \sum_i \varphi(\tau_r(f_r \cdot \gamma_i^{-1})) \cdot \pi\gamma_i$$

for $\varphi \in \mathrm{Hom}(F_r, D)$.

Note that this operator depends on the choice of τ and on the choice of coset representatives for $\Gamma\pi\Gamma$. We will see in the proof of the following lemma, however, that its restriction to $\mathrm{Hom}_\Gamma(F_r, D)$ is independent of the choice of coset representatives.

Lemma 3.2. *The operator $U : \mathrm{Hom}(F_r, D) \to \mathrm{Hom}(F_r, D)$ induces a map of chain complexes $U : C^r(\Gamma, D) \to C^r(\Gamma, D)$ and hence a map of cohomology groups $H^r(\Gamma, D) \to H^r(\Gamma, D)$.*

Proof. We first check that the action of U on an element of $C^r(\Gamma, D)$ does not depend on the choice of coset representatives for $\Gamma\pi\Gamma$. So assume for each i that we have $\Gamma\pi\gamma_i = \Gamma\pi\hat{\gamma}_i$ and write $\eta_i\pi\gamma_i = \pi\hat{\gamma}_i$ with $\eta_i \in \Gamma$. Note that $\pi^{-1}\eta_i^{-1}\pi = \hat{\gamma}_i\gamma_i^{-1}$ and thus is in Δ. We then have for $\varphi \in C^r(\Gamma, D) = \mathrm{Hom}_\Gamma(F_r, D)$,

$$\begin{aligned}\sum_i \varphi(\tau_r(f_r \cdot \hat{\gamma}_i^{-1})) \cdot \pi\hat{\gamma}_i &= \sum_i \varphi(\tau_r(f_r \cdot \gamma_i^{-1}\pi^{-1}\eta_i^{-1}\pi)) \cdot \eta_i\pi\gamma_i \\ &= \sum_i \varphi(\tau_r(f_r \cdot \gamma_i^{-1}) \cdot_\pi \pi^{-1}\eta_i^{-1}\pi) \cdot \eta_i\pi\gamma_i \\ &= \sum_i \varphi(\tau_r(f_r \cdot \gamma_i^{-1}) \cdot \eta_i^{-1}) \cdot \eta_i\pi\gamma_i \\ &= \sum_i \varphi(\tau_r(f_r \cdot \gamma_i^{-1})) \cdot \pi\gamma_i\end{aligned}$$

which establishes the independence.

We now verify that U induces a map from $C^r(\Gamma, D)$ to $C^r(\Gamma, D)$; that is, we must verify that if φ is invariant under Γ, then so is $\varphi\big|U$. We have

$$\begin{aligned}((\varphi\big|U) \cdot \gamma)(f_r) &= \big((\varphi\big|U)(f_r \cdot \gamma^{-1})\big) \cdot \gamma \\ &= \sum_i \varphi(\tau_r(f_r \cdot \gamma^{-1}\gamma_i^{-1})) \cdot \pi\gamma_i\gamma \\ &= \sum_i \varphi(\tau_r(f_r \cdot \gamma_i^{-1})) \cdot \pi\gamma_i = (\varphi\big|U)(f_r).\end{aligned}$$

Here the second to last equality follows from the independence of coset representatives established above as $\Gamma\pi\Gamma = \cup_i \Gamma\pi\gamma_i = \cup_i \Gamma\pi\gamma_i\gamma$.

Lastly, the fact that U commutes with d is immediate from its definition. □

The following simple lemma forms the basis of our argument.

Lemma 3.3. *Assume that D has a decreasing $R[S]$-filtration $\{F^n D\}$ satisfying hypothesis (1) of Theorem 3.1. Then*

$$\varphi \in C^r(\Gamma, F^n D) \quad \textit{implies} \quad \varphi\big|U \in C^r(\Gamma, F^{n+1} D).$$

Proof. We have

$$(\varphi\big|U)(f) = \sum_i \varphi(\tau(f \cdot \gamma_i^{-1})) \cdot \pi\gamma_i$$

which is in $F^{n+1}D$ as φ takes values in $F^n D$ and, by (1), $F^n D \cdot \pi \subseteq F^{n+1} D$. □

Lemma 3.4. *Assume that D has a decreasing $R[S]$-filtration $\{F^n D\}$ satisfying hypotheses (1) and (2) of Theorem 3.1.*

If Ψ is in the kernel of $H^r(\Gamma, D) \to H^r(\Gamma, D/F^0 D)$ and

$$\Psi \big| U = \alpha \Psi \quad \textit{with} \quad \alpha \in (R/\operatorname{Ann}_R(\Psi))^\times,$$

then $\Psi = 0$. That is, there are no ordinary Hecke-eigenclasses in the kernel of $H^r(\Gamma, D) \to H^r(\Gamma, D/F^0 D)$.

Proof. Since α is a unit modulo $\operatorname{Ann}_R(\Psi)$, there is an element $\beta \in R$ such that $\alpha\beta \equiv 1 \pmod{\operatorname{Ann}_R(\Psi)}$. We then have

$$\Psi = \beta^n \Psi \big| U^n$$

for all $n \geq 0$. If $\hat{\Psi}$ is a cocycle in $Z^r(\Gamma, F^0 D)$ representing Ψ, by Lemma 3.3, $\hat{\Psi} \in Z^r(\Gamma, F^n D) + B^r(\Gamma, D)$ for all $n \geq 0$. Thus, the image of Ψ in $H^r(\Gamma, D/F^n D)$ vanishes for all $n \geq 0$. By hypothesis (2) on $\{F^n D\}$, we have

$$H^r(\Gamma, D) = \varprojlim_n H^r(\Gamma, D/F^n D)$$

and thus $\Psi = 0$. □

We are now prepared to prove our main theorem.

Proof of Theorem 3.1. Let $\hat{\varphi}_0 \in Z^r(\Gamma, D/F^0 D)$ denote a cocycle representing $\varphi \in H^r(\Gamma, D/F^0 D)$ and let $\tilde{\varphi}_0 \in C^r(\Gamma, D)$ denote an arbitrary lift of $\hat{\varphi}_0$. Note that $d\tilde{\varphi}_0$ takes values in $F^0 D$ as $\hat{\varphi}_0$ is a cocycle.

Since φ is an ordinary U-eigenclass, there is some $\beta \in R$ be such that $\alpha\beta \equiv 1 \pmod{\operatorname{Ann}_R(\varphi)}$. Define

$$\tilde{\varphi}_n := \beta^n \tilde{\varphi}_0 \big| U^n \in C^r(\Gamma, D).$$

We claim that the image of $\tilde{\varphi}_n$ in $C^r(\Gamma, D/F^n D)$ is a cocycle. Indeed,

$$d\tilde{\varphi}_n = \beta^n d(\tilde{\varphi}_0 \big| U^n) = \beta^n (d\tilde{\varphi}_0) \big| U^n$$

which by Lemma 3.3 takes values in $F^n D$ as $d\tilde{\varphi}_0$ takes values in $F^0 D$.

Let $\hat{\varphi}_n \in Z^r(\Gamma, D/F^n D)$ denote the reduction of $\tilde{\varphi}_n$ modulo $F^n D$ and let φ_n denote the image of $\hat{\varphi}_n$ in $H^r(\Gamma, D/F^n D)$. We will show that for each $n > 0$, φ_n is U-eigenvector with eigenvalues α and that φ_n is a lift of φ_{n-1}.

For the first claim, since $\varphi_0 \big| U = \alpha \varphi_0$, we have $\tilde{\varphi}_0 \big| U - \alpha \tilde{\varphi}_0$ is in $C^r(\Gamma, F^0 D) + B^r(\Gamma, D)$. Thus

$$\tilde{\varphi}_n | U - \alpha \tilde{\varphi}_n = \beta^n \tilde{\varphi}_0 \big| U^{n+1} - \alpha \beta^n \tilde{\varphi}_0 \big| U^n = \beta^n \left(\tilde{\varphi}_0 \big| U - \alpha \tilde{\varphi}_0 \right) \big| U^n$$

which, by Lemma 3.3, is in $C^r(\Gamma, F^n D) + B^r(\Gamma, D)$. Therefore, $\varphi_n \big| U = \alpha \varphi_n$.

For the second claim, we have

$$\tilde{\varphi}_n - \tilde{\varphi}_{n-1} = \beta^n \tilde{\varphi}_0 \big| U^n - \beta^{n-1} \tilde{\varphi}_0 \big| U^{n-1} = \beta^{n-1} \left(\beta \tilde{\varphi}_0 \big| U - \tilde{\varphi}_0 \right) \big| U^{n-1}$$

which, by Lemma 3.3, is in $C^r(\Gamma, F^nD) + B^r(\Gamma, D)$ as $\beta\varphi_0\big|U = \varphi_0$.

Thus,

$$\{\varphi_n\} \in \varprojlim_n H^r(\Gamma, D/F^nD) \cong H^r(\Gamma, \varprojlim_n D/F^nD) \cong H^r(\Gamma, D) \qquad (1)$$

where the last equality follows from hypothesis (2) on $\{F^nD\}$. The collection of classes $\{\varphi_n\}$ therefore corresponds to a single class $\Phi \in H^r(\Gamma, D)$. It is immediate that Φ lifts φ and that $\Phi\big|U = \alpha\Phi$.

The equality of the R-annihilators of Φ and φ and the uniqueness of Φ now follow from Lemma 3.4. Indeed, let Φ' in $H^r(\Gamma, D)$ be any ordinary U-eigenlift of φ with $\Phi'\big|U = \alpha'\Phi'$. Then $\mathrm{Ann}_R(\Phi') \subseteq \mathrm{Ann}_R(\varphi)$ as Φ' maps onto φ. Conversely, if $x \in \mathrm{Ann}_R(\varphi)$, then $x\Phi'$ maps to 0 in $H^r(\Gamma, D/F^0D)$, while $(x\Phi')\big|U = \alpha'(x\Phi')$. Since $\mathrm{Ann}_R(x\Phi') \supseteq \mathrm{Ann}_R(\Phi')$, α' is a unit modulo $\mathrm{Ann}_R(x\Phi')$. Thus, by Lemma 3.4, $x\Phi' = 0$ and so $\mathrm{Ann}_R(\Phi') = \mathrm{Ann}_R(\varphi)$.

Finally, since U scales Φ' by α' and φ by α, it is automatic that $\alpha' - \alpha$ is in $\mathrm{Ann}_R(\varphi) = \mathrm{Ann}_R(\Phi')$. In particular, Φ' also satisfies $\Phi'\big|U = \alpha\Phi'$. Thus, the difference $\Phi - \Phi'$, which maps to 0 in $H^r(\Gamma, D/F^0D)$, also scales by α under U. Since $\mathrm{Ann}_R(\Phi - \Phi')$ contains $\mathrm{Ann}_R(\Phi) = \mathrm{Ann}_R(\Phi')$, α is again a unit modulo $\mathrm{Ann}_R(\Phi - \Phi')$ and so, by Lemma 3.4, we deduce that $\Phi = \Phi'$. □

Remark 3.5. If R is assumed to be a local ring, we can strengthen the uniqueness claim of Theorem 3.1. Indeed, in this case, any eigenlift of an ordinary eigenclass is automatically ordinary. To see this, let $\varphi \in H^r(\Gamma, D/F^0D)$ be an ordinary U-eigenvector with $\varphi\big|U = \alpha\varphi$. If $\Phi \in H^r(\Gamma, D)$ is an eigenlift of φ with $\Phi\big|U = \alpha'\Phi$, then $\alpha' \equiv \alpha \pmod{\mathrm{Ann}_R(\varphi)}$. Thus α' is a unit modulo $\mathrm{Ann}_R(\varphi)$, which is a proper ideal of R since $\varphi \neq 0$. Thus since R is a local ring, we deduce that α' is a unit in R.

4 Filtrations and liftings

We return now to the setting of section 2. In order to invoke the results of the previous section, we introduce a filtration on $\mathbf{D}_\lambda$ which satisfies the hypotheses of Theorem 3.1.

4.1 An S-stable filtration on $\mathbf{D}_\lambda$

We define a filtration on $\mathbf{D}_\lambda$ as follows. For $N \in \mathbb{Z}^{\geq 0}$, set

$$\underline{\mathrm{Fil}}^N\mathbf{D}_\lambda := \left\{ \mu \in \mathbf{D}_\lambda \ : \ \mathrm{ord}_p\left(\mu(f_{rst})\right) \geq \left\lceil \frac{N-(r+s+t)}{2} \right\rceil \text{ for } r,s,t \geq 0 \right\}$$

where $\lceil x \rceil$ is the smallest integer greater than or equal to x. (We note that this filtration is too large to satisfy the hypotheses of Theorem 3.1. In the following subsection, we will replace it by a slightly smaller filtration.)

Proposition 4.1. *The filtration $\underline{\mathrm{Fil}}^N\mathbf{D}_\lambda$ is stable under the action of S.*

Proof. By definition, $(\mu|\gamma)(f_{rst}) = \mu(\gamma f_{rst})$. Let $\lambda = (k_1, k_2, k_3)$. By Lemma 2.1, for $\gamma \in S'$, we have

$$(\gamma f_{rst})\left(\begin{smallmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{smallmatrix}\right) =$$
$$\det(\gamma)^{k_3}(a_{12}+a_{22}x+a_{32}y)^r(a_{13}+a_{23}x+a_{33}y)^s(-m_{32}+m_{12}y+m_{22}z-m_{12}xz)^t$$
$$(a_{11}+a_{21}x+a_{31}y)^{k_1-k_2-r-s}(m_{33}-m_{13}y-m_{23}z+m_{13}xz)^{k_2-k_3-t}.$$

We expand out the last terms with the binomial theorem (keeping in mind that the exponents may be negative). We have

$$\begin{aligned}
&(\gamma f_{rst})\left(\begin{smallmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{smallmatrix}\right) \\
&\quad = \det(\gamma)^{k_3}(a_{12}+a_{22}x+a_{32}y)^r \cdot \\
&\quad (a_{13}+a_{23}x+a_{33}y)^s(-m_{32}+m_{12}y+m_{22}z-m_{12}xz)^t \cdot \\
&\quad (a_{11})^{k_1-k_2-r-s}\left(\sum_{j=0}^{\infty}\binom{k_1-k_2-r-s}{j}\left(\frac{a_{21}}{a_{11}}x+\frac{a_{31}}{a_{11}}y\right)^j\right) \cdot \\
&\quad (m_{33})^{k_2-k_3-t}\left(\sum_{j=0}^{\infty}\binom{k_2-k_3-t}{j}\left(-\frac{m_{13}}{m_{33}}y-\frac{m_{23}}{m_{33}}z+\frac{m_{13}}{m_{33}}xz\right)^j\right) \\
&\quad = \sum_{a,b,c}\alpha_{abc}\, x^a y^b z^c.
\end{aligned}$$

Binomial coefficients $\binom{a}{b}$ with $a \geq 0$ and $b > a$ should be interpreted as 0. Note that a_{11}, a_{22}, a_{33}, m_{22}, and m_{33} are all units while a_{21}, a_{31}, a_{32}, m_{12}, m_{13}, and m_{23} are all multiples of p.

We now consider the possible valuations of α_{abc} for a fixed triple (a, b, c). Note that the coefficient of a monomial of degree d appearing in the expansion of the first three nonconstant factors of this product has valuation at least $d - (r + s + t)$. (This follows as the only non-linear term that appears in these factors has a coefficient divisible by p.) Also note that the coefficient of a monomial of degree m in the expansion of the last two nonconstant factors will have valuation at least equal to $\left\lceil \frac{m}{2} \right\rceil$. (This follows as all coefficients that appear in these expressions are divisible by p including the coefficient of the xz term.) Thus, the valuation of α_{abc} is at least $\left\lceil \frac{(a+b+c)-(r+s+t)}{2} \right\rceil$.

Note that $\mu(\gamma f_{rst}) = \sum_{a,b,c} \alpha_{a,b,c}\mu(f_{abc})$ and we wish to show that this sum has valuation at least $\left\lceil \frac{N-(r+s+t)}{2} \right\rceil$. We do this term-by-term; fix a triple (a, b, c). If $a + b + c \geq N$, then by the above computation, we have that the valuation of α_{abc} is large enough. If $a + b + c < N$, then, as μ is in $\underline{\mathrm{Fil}}^N \mathbf{D}_\lambda$, we have that $\mu(f_{abc})$ has valuation at least $\left\lceil \frac{N-(a+b+c)}{2} \right\rceil$. Since

$$\left\lceil \frac{N-(a+b+c)}{2} \right\rceil + \left\lceil \frac{(a+b+c)-(r+s+t)}{2} \right\rceil \geq \left\lceil \frac{N-(r+s+t)}{2} \right\rceil$$

we have that $\mu|\gamma$ is in $\underline{\mathrm{Fil}}^N \mathbf{D}_\lambda$.

Lastly, we consider the action by the element π. In this case,

$$(\mu|\pi)(f_{rst}) = \mu(\pi f_{rst}) = p^{r+2s+t}\mu(f_{rst}).$$

Thus, if μ is in $\underline{\mathrm{Fil}}^N \mathbf{D}_\lambda$ so is $\mu|\gamma$. □

Lemma 4.2. *If $\mu \in \underline{\mathrm{Fil}}^N \mathbf{D}_\lambda$ and $\mu(f_{000}) = 0$, then $\mu|\pi \in \underline{\mathrm{Fil}}^{N+1} \mathbf{D}_\lambda$.*

Proof. Since $(\mu|\pi)(f_{rst}) = p^{r+2s+t}\mu(f_{rst})$, if $(r, s, t) \neq (0, 0, 0)$, then

$$\mathrm{ord}_p\left((\mu|\pi)(f_{rst})\right) > \mathrm{ord}_p\left(\mu(f_{rst})\right).$$

Moreover, if $\mu(f_{000}) = 0$, then $(\mu|\pi)(f_{000}) = 0$. Thus, $\mu \in \underline{\mathrm{Fil}}^N \mathbf{D}_\lambda$ implies that $\mu|\pi \in \underline{\mathrm{Fil}}^{N+1} \mathbf{D}_\lambda$. □

4.2 The main filtration

Recall the map $\rho_\lambda : \mathbf{D}_\lambda \to L_\lambda$ given by evaluation at the function f_λ sending $g \in G(\mathbb{O}_p)$ to $f_\lambda(g) = v_\lambda \cdot g$. Set $K_\lambda := \ker(\rho_\lambda)$ which is an S-submodule as ρ_λ is S-equivariant. We then define our main filtration on $\mathbf{D}_\lambda$ by

$$\mathrm{Fil}^N \mathbf{D}_\lambda := \underline{\mathrm{Fil}}^N \mathbf{D}_\lambda \cap K_\lambda.$$

Before checking that this filtration satisfies the hypotheses of Theorem 3.1, we introduce one lemma.

Lemma 4.3. *If $\mu \in K_\lambda$ then $\mu(f_{000}) = 0$.*

Proof. Let v_λ, v_2, $\ldots$, v_d be a basis of $V_\lambda(\mathbb{Q}_p)$. Since v_λ is a highest weight vector for N^{opp}, we have

$$v_\lambda \cdot \left(\begin{smallmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{smallmatrix}\right) = v_\lambda + \sum_{i=2}^{d} P_i(x, y, z) v_i.$$

Thus,

$$\rho_\lambda(\mu) = \mu(f_{000}) v_\lambda + \sum_{i=2}^{d} \mu(P_i') v_i$$

where P_i' is the unique extension of P_i to X. Therefore, $\mu(f_{000}) = 0$ as μ is in the kernel of specialization. □

Proposition 4.4. *We have that $\{\mathrm{Fil}^N \mathbf{D}_\lambda\}$ is a decreasing $\mathbb{Z}_p[S]$-stable filtration such that*

1. *$\mathrm{Fil}^N \mathbf{D}_\lambda|\pi \subseteq \mathrm{Fil}^{N+1} \mathbf{D}_\lambda$ for all $N \geq 0$,*
2. *the natural map $\mathbf{D}_\lambda \to \varprojlim \mathbf{D}_\lambda / \mathrm{Fil}^N \mathbf{D}_\lambda$ is an isomorphism.*

Proof. The first part of the proposition follows from Lemma 4.2 and Lemma 4.3. For the second part, it is clear from the definitions that $\cap \operatorname{Fil}^N \mathbf{D}_\lambda = 0$. Conversely, if $\{\mu_N\} \in \varprojlim \mathbf{D}_\lambda / \operatorname{Fil}^N \mathbf{D}_\lambda$, for a fixed triple (a, b, c) the sequence $\{\mu_N(f_{abc})\}$ is Cauchy converging to say $\alpha_{abc} \in \mathbb{Z}_p$. Let μ be the unique distribution such that $\mu(f_{abc}) = \alpha_{abc}$ for all a, b, c. Then μ projects to μ_N for each $N \geq 0$. □

4.3 Lifting Hecke-eigensymbols

Let $\mathcal{H}$ denote the abstract Hecke-algebra generated over $\mathbb{Z}_p$ by U_p and by $T(\ell, k)$ for all primes $\ell \neq p$ and $k = 1, 2, 3$. Recall that the specialization map

$$\rho_k^r : H^r(\Gamma_0, \mathbf{D}_\lambda) \to H^r(\Gamma_0, L_\lambda)$$

is $\mathcal{H}$-equivariant. We offer the following theorem (which is implied by Theorem 2.6).

Theorem 4.5. *If $\varphi \in H^r(\Gamma_0, L_\lambda)$ is an $\mathcal{H}$-eigenvector whose U_p-eigenvalue is a unit, then there exists a unique $\mathcal{H}$-eigenvector $\Phi \in H^r(\Gamma_0, \mathbf{D}_\lambda)$ that specializes to φ.*

Proof. By construction, $\mathbf{D}_\lambda / F^0 \mathbf{D}_\lambda \cong L_\lambda$. Thus, Theorem 3.1 applies and there exists a unique U_p-eigenvector Φ lifting φ (see Remark 3.5). Moreover, Φ is automatically an $\mathcal{H}$-eigenvector. Indeed, if $T \in \mathcal{H}$ and $\varphi \big| T = a\varphi$, then, since T and U_p commute, both $\Phi \big| T$ and $a\Phi$ are U_p-eigenlifts of $a\varphi$. By uniqueness, we conclude that $\Phi \big| T = a\Phi$. □

Remark 4.6. The above argument works equally well if one had a $\mathcal{H}$-eigenvector $v \in H^r(\Gamma_0, L_\lambda) \otimes \mathcal{O}$ for some finite extension $\mathcal{O}$ of $\mathbb{Z}_p$.

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