EVERYWHERE UNRAMIFIED AUTOMORPHIC COHOMOLOGY FOR $SL(3, \mathbb{Z})$

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Abstract. We conjecture that the only irreducible cuspidal automorphic representations for GL(3)/$\mathbb{Q}$ of cohomological type and level 1 are (up to twisting) the symmetric square lifts of classical cuspforms on GL(2)/$\mathbb{Q}$ of level 1. We present computational evidence for this conjecture.

1. Statement and explanation of a conjecture

Arithmetic objects defined over $\mathbb{Q}$ and unramified everywhere are rare. For example, a theorem of Minkowski states that there are no finite extensions of $\mathbb{Q}$ unramified everywhere. A theorem of Fontaine [8] states that there are no abelian varieties defined over $\mathbb{Q}$ unramified everywhere.

In this note we consider automorphic cohomology for $\mathrm{SL}(n, \mathbb{Z})$ defined over $\mathbb{Q}$ and unramified everywhere, $n \geq 2$. Let $\mathbb{A}$ denote the adeles of $\mathbb{Q}$. Given a Hecke-eigenclass class α in the cuspidal cohomology of $SL(n, \mathbb{Z})$ with coefficients in an irreducible finite dimensional complex rational representation V of $\mathrm{GL}(n, \mathbb{C})$, there is uniquely determined an irreducible cuspidal automorphic representation $\pi_\alpha = \otimes \pi_{\alpha,v}$ of $\mathrm{GL}(n, \mathbb{A})$. The infinity type $\pi_{\alpha,\infty} = \phi(V)$ depends only on V and for every prime p, $\pi_{\alpha,p}$ is an irreducible unramified principle series representation of $GL(n, \mathbb{Q}_p)$ depending only on the Hecke eigenvalues of α at p. We say an irreducible cuspidal automorphic π is cohomological if $\pi_\infty = \phi(V)$ for some V as above. Then each cohomological π corresponds to some such Hecke-eigenclass α. See [7]. (The cuspidal cohomology is defined to be the subspace of the cohomology which can be represented by cuspidal automorphic differential forms on the appropriate symmetric space.)

We ask: are there no nonzero cuspidal cohomological automorphic representations of $GL(n)/\mathbb{Q}$ unramified everywhere, or equivalently, is $H^*_{cusp}(SL(n, \mathbb{Z}), V) = 0$ for all V.

The answer is well-known to be "no" for $n = 2$. For then the theory of classical modular forms and the Shimura-Eichler theorem combine to tell us that if $V = \mathrm{Sym}^g(\mathbb{C}^2)$, then $H^*_{cusp}(SL(2, \mathbb{Z}), V) \neq 0$ if g is even and either $= 10$ or ≥ 14. For example, Ramanujan's Δ gives rise to a nontrivial class in $H^*_{cusp}(SL(2, \mathbb{Z}), \mathrm{Sym}^{10}(\mathbb{C}^2))$. So although arithmetic objects defined over $\mathbb{Q}$ unramified everywhere are rare, there are some. It should be noted that the existence of cuspforms of level 1 for SL(2,Z) can be accounted for by the simple topological fact that the Euler characteristic of $\mathrm{SL}(2, \mathbb{Z})$ is non-zero, or by the representation-theoretic fact that $\mathrm{GL}(2, \mathbb{R})$ has a

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nontrivial discrete series, whereas neither of the analogous statements is true for GL(n) if $n \geq 3$.

Moreover, certain instances of Langlands' functoriality would lift everywhere unramified $GL(2)$-representations to $GL(n)$-representations, $n \geq 3$, again everywhere unramified. If the infinity type lifts to a cohomological infinity type, one would obtain more examples of automorphic cohomology unramified everywhere. In particular, the symmetric-square lift of [10] (a proven instance of Langlands' functoriality) produces automorphic cohomology unramified everywhere for $GL(3)$. For more details see [4]. However, these lifted classes are in a sense not "brand new".

Let $S(V) \subset H^*_{cusp}(\mathrm{SL}(3,\mathbb{Z}), V)$ be the $\mathbb{C}$-span of the set of $\alpha \in H^*_{cusp}(\mathrm{SL}(3,\mathbb{Z}), V)$ such that π_α is a symmetric-square lift. The purpose of this note is to state the following conjecture and present some computational evidence towards it.

Conjecture 1.1. $H^*_{cusp}(\mathrm{SL}(3,\mathbb{Z}), V) = S(V)$.

Remarks:

1. We say that V is essentially selfdual if it is isomorphic to the tensor product of its contragredient with some character. Theorem II.6.12 of [6] says that the cuspidal cohomology vanishes if V is not essentially selfdual. Let W_g be the irreducible representation of $\mathrm{GL}(3,\mathbb{C})$ of highest weight $(2g, g, 0)$. The only essentially selfdual V are of the form $W_g \otimes \det^b$ for some $b \in \mathbb{Z}$ and some $g \geq 0$. Thus $H^*_{cusp}(\mathrm{SL}(3,\mathbb{Z}), V) = 0$ unless V is of this form. Also $S(V) = 0$ unless $V \cong W_g \otimes \det^b$. Since det is trivial on $\mathrm{SL}(3,\mathbb{Z})$, Conjecture 1 is equivalent to the statement that $H^*_{cusp}(SL(3,\mathbb{Z}), W_g) = S(W_g)$ for all $g \geq 0$.

2. Classical cuspforms of level 1 occur only in weights 12,16,18,.... The symmetric square of a classical cuspform of level 1 and weight k gives rise to a nonzero class in $H^*_{cusp}(SL(3,\mathbb{Z}), W_{k-2})$.

3. It follows from Langlands' hypothesis, as made precise in Question 4.16 of [7] that Conjecture 1 would imply that if g is a positive integer there is no 3-dimensional motive defined over $\mathbb{Q}$ with Hodge numbers $(0, 2g)$,(g, g),$(2g, 0)$ and unramified everywhere except the symmetric-square of a 2-dimensional motive attached to a classical cusp forms of level 1.

4. It follows from the Fontaine-Mazur conjecture [9] and remark 3 that if g is a positive integer there is no 3-dimensional p-adic representation of the absolute Galois group of $\mathbb{Q}$ crystalline at p with Hodge-Tate numbers $0, g, 2g$ and unramified outside p, ∞ except the symmetric-square of a 2-dimensional Galois representation attached to a classical cusp form of level 1.

5. If we fix $V = \mathbb{C}$ and allow n to vary instead, there is a theorem of Steve Miller [12] that says that $H^*_{cusp}(SL(n,\mathbb{Z}), \mathbb{C}) = 0$ if $n < 23$. It is not known what happens if n is larger.

We computed and found that $H^*_{cusp}(SL(3,\mathbb{Z}), W_g) = S(W_g)$ for all g (even or odd) up to $g = 120$. In the next section, we will explain how we did the calculations and what checks we have on them.

It remains to explain the numerology according to which it seems reasonable to make our conjecture based on this evidence. Find the value of g which makes the following analogy convincing: 11 is to 53 as 12 is to g. Explanation: For $GL(2)$, the smallest level N for which $H^*_{cusp}(\Gamma_0(N), \mathbb{C}) \neq 0$ is $N = 11$. Whereas the smallest weight k for which $H^*_{cusp}(SL(2,\mathbb{Z}), \mathrm{Sym}^{k-2}(\mathbb{C}^2)) \neq 0$ is $k = 12$. Moreover, computationally, these two problems are roughly of the same size, involving row-reduction of a matrix with n rows and m columns where both n, m are on the order of 11 (or 12).

Now by [3], for $GL(3)$, the smallest prime level N for which $H^*_{cusp}(\Gamma_0(N), \mathbb{C}) \neq 0$ is $N = 53$. (There are no symmetric-square lifts of prime level.) Computing this cohomology group involves row-reduction of a matrix with n rows and m columns where both n, m are on the order of 53^2. So one might expect that the smallest weight g for which $H^*_{cusp}(SL(3,\mathbb{Z}), W_g) \neq S(W_g)$ should be a value where the computation of the group is about the same size. Now, computing $H^*_{cusp}(SL(3,\mathbb{Z}), W_g)$ involves row-reduction of a matrix with n rows and m columns where both n, m are on the order of g^3.

We then have the heuristic that $g^3 \approx 53^2$ or $g \approx 14$. Since no cuspidal cohomology showed up even for much larger g, we are willing to make Conjecture 1.

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2. Computations and Results

It is shown in [3] that $\mathrm{H}^3(\mathrm{SL}(3,\mathbb{Z}), V) = \mathrm{H}^3_{cusp}(\mathrm{SL}(3,\mathbb{Z}), V) \oplus$boundary cohomology. Although stated there only for trivial coefficients, the proof remains valid for V coefficients. If we let $s_k(1)$ denote the dimension of the space of classical holomorphic cuspforms of weight k and level $\mathrm{SL}_2(\mathbb{Z})$, then it follows from [1, Thm. 3.2] that for even g the dimension of the boundary cohomology is at least $2s_{g+2}(1) + 1$. Since $\dim_{\mathbb{C}} S(W_g) = s_{g+2}(1)$ by the multiplicity 1 theorem for automorphic representations of GL(3) we see

Theorem 2.1. *Let $g \geq 0$ be an even integer. Then*

$$\dim \mathrm{H}^3(\mathrm{SL}(3,\mathbb{Z}), W_g) \geq 3s_{g+2}(1) + 1$$

and if equality holds we have $\mathrm{H}^3_{cusp}(\mathrm{SL}(3,\mathbb{Z}), W_g) = S(W_g)$.

It is more efficient to carry out our calculations over fields of relatively small positive characteristic rather than over $\mathbb{C}$. Choose a prime p and let $\mathbb{F}_p$ be the finite field of size p. For each g we choose once and for all a $\mathbb{Z}$-lattice L_g in W_g stable under $\mathrm{GL}(3,\mathbb{Z})$, and we define $W_g(\mathbb{F}_p)$ to be the reduction of L_g mod p. If $p > 2g$ then $W_g(\mathbb{F}_p)$ is irreducible and is, up to isomorphism, independent of the choice of L_g. We insist henceforth that $p > 2g$. It follows from the Universal Coefficients Theorem that $\dim_{\mathbb{C}} \mathrm{H}^3(\mathrm{SL}(3,\mathbb{Z}), W_g) \leq \dim_{\mathbb{F}_p} \mathrm{H}^3(\mathrm{SL}(3,\mathbb{Z}), W_g(\mathbb{F}_p))$. We then see from the above theorem that

Theorem 2.2. *Let $p > 2g$ be prime. Then*

(1) *If g is even and*

$$\dim_{\mathbb{F}_p} \mathrm{H}^3(\mathrm{SL}_3(\mathbb{Z}), W_g(\mathbb{F}_p)) = 3s_{g+2}(1) + 1$$

then $\mathrm{H}^3_{cusp}(\mathrm{SL}(3,\mathbb{Z}), W_g) = S(W_g)$.

(2) *If g is odd and*

$$\dim_{\mathbb{F}_p} \mathrm{H}^3(\mathrm{SL}_3(\mathbb{Z}), W_g(\mathbb{F}_p)) = 0$$

then $\mathrm{H}^3_{cusp}(\mathrm{SL}(3,\mathbb{Z}), W_g) = S(W_g) = 0$.

So to prove *non-existence* of cuspidal cohomology, it suffices to work over finite fields $\mathbb{F}_p$.

The computations of $\mathrm{H}^3(\mathrm{SL}(3,\mathbb{Z}), W_g(\mathbb{F}_p))$ were carried out by a refinement of the method developed in [3, 1, 2]. In particular, we assume $p > 3$ and we compute homology groups rather than cohomology groups and use the natural duality as in [5, Section 7].

Thus we wish to calculate $\mathrm{H}_3(\mathrm{SL}_3(\mathbb{Z}), W_g(\mathbb{F}_p))$. The broad outline of our calculations follows that of [1]. Given an $\mathbb{F}_p$ representation V of $\mathrm{GL}(3,\mathbb{Z})$, we first use a slight modification of their Theorem 1 to identify $\mathrm{H}_3(\mathrm{SL}_3(\mathbb{Z}), V)$ with the subspace of all $v \in V$ such that

(1) $v \cdot \sigma = \epsilon(\sigma) v$ for all monomial matrices $\sigma \in \mathrm{SL}_3(\mathbb{Z})$, where $\epsilon(\sigma) = \pm 1$ is the sign of the permutation induced by σ.
(2) $v + v \cdot h + v \cdot (h^2) = 0$,

where

$$h = \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We will refer to condition 1 as the "semi-invariant condition" and to condition 2 as the "h-condition". Given a sufficiently concrete realization of V, computationally identifying the subspace of V satisfying these conditions is simply an exercise in linear algebra. The difficulty arises from the facts that the dimensions of the spaces $V = W_g$ that we consider are quite large, and that computing the action of an element of $\mathrm{SL}_3(\mathbb{Z})$ on a vector in V can be time consuming. For $g = 120$, the dimension of W_g is $(g+1)^3 = 1771561$, and computing the action of an arbitrary element of $\mathrm{SL}_3(\mathbb{Z})$ on an element of W_g can involve computing the 120th power of a polynomial in 3 variables.

2.1. **A model for W_g.** We'll now turn to the model of the right GL(3)-module $W_g(\mathbb{F}_p)$ with which we are computing. To ease notation let $\mathbb{F} = \mathbb{F}_p$. Let V_g be $\mathrm{Sym}_g(\mathbb{F}^3)$ and V_g^* its contragredient. We view V_g as the space of homogeneous polynomials of degree g in the variables x, y, z and V_g^* as homogenous polynomials in the their duals, $\hat{x}, \hat{y}, \hat{z}$. Then $W_g(\mathbb{F})$ naturally sits as a quotient of $V_g \otimes V_g^* \otimes \det^g$. Since $p > 2g$, $W_g(F_p)$ actually splits off as a submodule, which we will denote by W. Indeed W is the submodule generated by the vector $w_g = x^g \otimes \hat{z}^g \in V_g \otimes V_g^* \otimes \det^g$.

Given an element v of W, we say that v is a vector of weight (a_1, a_2, a_3), $0 \leq a_i \leq p-1$, if

$$v\,\mathrm{diag}(\alpha, \beta, \gamma) = \alpha^{a_1} \beta^{a_2} \gamma^{a_3}$$

for all $\alpha, \beta, \gamma \in \mathbb{F}$. Note that this is equivalent to saying that for any monomial term $x^{m_1} y^{m_2} z^{m_3} \otimes \hat{x}^{n_1} \hat{y}^{n_2} \hat{z}^{n_3}$ appearing in v, $m_i - n_i$ is congruent to a_i mod $p-1$ and since $p > 2g \geq |m_i - n_i|$, this last condition is equivalent to $m_i - n_i = a_i$. We say that a weight $\lambda = (a_1, a_2, a_3)$ is *dominant* if $a_1 \geq a_2 \geq a_3$, and we say λ is *regular* if all the a_i are distinct. If $\lambda' = (a_1', a_2', a_3')$ is another weight then we say $\lambda' < \lambda$ if $\lambda - \lambda' = (a_1 - a_1', a_2 - a_2', a_3 - a_3')$ is dominant.

If $v \in W$ has a decomposition $v = \sum_i v_{\lambda_i}$ in $V_g \otimes V_g^*$ as a sum of vectors with distinct weights λ_i then each v_{λ_i} must lie in W. We write W_λ for the subspace of vectors of weight λ in W. Note that the generating vector w_g above sits in $W_{(2g,g,0)}$.

Recall that the representation W is essentially selfdual. In particular, if we let $\rho : V_g \otimes V_g^* \otimes \det^g \to V_g \otimes V_g^* \otimes \det^g$ be the linear map defined on pure tensors by

$$\rho(P(x,y,z) \otimes Q(\hat{x},\hat{y},\hat{z})) = Q(x,y,z) \otimes P(\hat{x},\hat{y},\hat{z})$$

then ρ preserves W and satisfies

$$\rho(v\gamma) = \rho(v)^t\gamma^{-1}\det(\gamma)^{2g}.$$

So ρ exhibits the isomorphism of W with $W^* \otimes \det^{2g}$. Note that ρ sends $W_{(a_1,a_2,a_3)}$ bijectively to $W_{(2g-a_1,2g-a_2,2g-a_3)}$.

We now want to find a basis for W inside $V_g \otimes V_g^* \otimes \det^g$. Indeed we will find bases of each of W's weight spaces W_λ. Let

$$E_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

If $w \in W_\lambda$ for some weight λ then both wE_1 and wE_2 are in $w + \sum_{\lambda'<\lambda} W_{\lambda'}$. Moreover, since w_g is a highest weight vector for W, the iterates of E_1 and E_2 applied to w_g span W. Note that we know the dimension of W is $(g+1)^3$ and we can use Kostant's weight multiplicity formula [11] to find the dimension m_λ of each weight space W_λ. This allows us to quickly recognize when we have performed enough iterations of the E_i without having to actually verify that the space we have generated is closed under these operators. (We did perform this verification for a few small values of g as a check on our programs.)

We can exploit the Weyl group of GL_3 for a small improvement. Let n be any representative of the Weyl automorphism sending $\mathrm{diag}\,(\alpha,\beta,\gamma)$ to $\mathrm{diag}\,(\gamma,\beta,\alpha)$. Let $M^>, M^=$ and $M^<$ be the sets of weights (a_1,a_2,a_3) with $a_1 > a_3$, $a_1 = a_3$ and $a_1 < a_3$, respectively. Then $\oplus_{\lambda\in M^<} W_\lambda$ is closed under the E_i, and is equal to $n\,(\oplus_{\lambda\in M^>} W_\lambda)\,n^{-1}$. Thus it suffices to find bases for W_λ with $\lambda \in M^> \cup M^=$, and we can do so without considering any vectors with weights in $M^<$.

Our algorithm is then as follows. For each weight $\lambda \in M^> \cup M^=$ we maintain a list of linearly independent vectors that we know lie in W_λ. We begin with w_g on our list for $W_{(2g,g,0)}$ and apply each of the E_i. We then decompose the images into a sum of weight vectors w_{λ_i}. If we have already found m_{λ_i} independent vectors in W_{λ_i} we discard w_{λ_i}. Otherwise, for $\lambda_i \in M^\geq$ we check if w_{λ_i} is independent from the vectors we've already found in W_{λ_i} and if so we add it to the list. We also add w_{λ_i} to our queue of vectors to which we need to apply the weight lowering operators. Finally if $\lambda_i \in M^>$ we add $nw_{\lambda_i}n^{-1}$ to a separate list of basis vectors for the spaces W_λ, $\lambda \in M^<$ that we maintain. We remark that we never found a linear dependence among the first m_{λ_i} vectors we produced in W_{λ_i}.

We then check to see if we have found $(g+1)^3$ linearly independent vectors in W yet. If so, we are finished, and if not we repeat the previous step replacing w_g by the next vector on our queue. We have observed empirically that building our queue up as above generates a basis in fewer steps than does choosing to act on a remaining vector of maximal weight at each stage.

Finally we note that at each stage of our algorithm we can produce up to $4g$ new vectors to consider, and so the number of linearly independent vectors in W we have found tends to grows in large steps with each iteration. The fact that we always do find a step that terminates with exactly $(g+1)^3$ vectors (and no more) provides a check on the calculations.

2.2. **Finding the semi-invariant vectors.** Having now chosen a basis of W, we wish to use it to find a basis of the space of the semi-invariant vectors in W. We easily generalize the results of Section 7.1 of [1] (which were not directly applicable to [2] since we were not working only in level 1). Let R be the group of monomial matrices in $\mathrm{SL}(3,\mathbb{Z})$, and let S be the 4-element subgroup of R consisting of diagonal matrices. Let P be a 6-element subgroup of R such that $R = S \rtimes P$. A vector $v \in W$ is semi-invariant if and only if $v\sigma = \epsilon(\sigma)v$ for $\sigma \in S$ and for $\sigma \in P$.

S preserves each weight space W_λ and we see

Lemma 2.3. *Let $\lambda = (a,b,c)$. Then S acts as the identity on W_λ if $a \equiv b \equiv c \equiv g \pmod 2$, and W_λ contains no nonzero vectors fixed by S otherwise.*

Proof. If a, b, and c are all congruent modulo 2 then S clearly acts as the identity on W_λ.

Conversely, suppose $0 \neq w \in W_\lambda$ is invariant by S. Then since $\mathrm{diag}\,(-1,-1,1) \in S$ acts on w by multiplication by $(-1)^{a+b}$ we see that $a \equiv b \pmod 2$. Likewise $b \equiv c \pmod 2$ as $\mathrm{diag}\,(1,-1,-1) \in S$. Then $\mathrm{diag}\,(-1,-1,-1)$ acts on w as $(-1)^{a+b+c} = (-1)^{3a}$. But $\mathrm{diag}\,(-1,-1,-1)$ is central in $\mathrm{GL}(3,\mathbb{F})$ and so acts on all of W as $(-1)^{3g}$. So $3g \equiv 3a \pmod 2$ and the lemma follows. □

For each dominant weight λ, let

$$W_{(\lambda)} = \sum_{\sigma \in P} W_{\sigma(\lambda)}$$

where $\sigma(a_1,a_2,a_3) = (a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$. Then P acts on each $W_{(\lambda)}$ and W is the sum of the $W_{(\lambda)}$. So to find a basis of the semi-invariants, it suffices to find bases of the subspaces of the $W_{(\lambda)}$ that are semi-invariant by P for each dominant λ all of whose coordinates are congruent mod 2.

If λ is a regular dominant weight made up of parts all congruent mod 2, and $b_1,\ldots,b_n$ is a basis of the weight space W_λ then the vectors

$$s_i = \sum_{\sigma \in P} b_i\sigma$$

form our desired basis of the semi-invariants in $W_{(\lambda)}$. For non-regular weights things are slightly more involved, since the s_i no longer need be linearly independent. They do still span the space of semi-invariants in $W_{(\lambda)}$ and so we throw out the redundant ones, obtaining a basis of their span.

Note now that if $\lambda = (a_1,a_2,a_3)$ is dominant then $\lambda' = (2g-a_3, 2g-a_2, 2g-a_1)$ is also dominant and ρ sends $W_{(\lambda)}$ bijectively to $W_{(\lambda')}$ and preserves semi-invariants. So it suffices to find bases of the semi-invariants of $W_{(\lambda)}$ for the dominant $\lambda = (a_1,a_2,a_3)$ such that $a_2 \geq g$. We then apply ρ to these bases to obtain bases for semi-invariants in the other $W_{(\lambda)}$. Finally, note that if λ is dominant and essentially selfdual (i.e. $a_2 = g$), $\rho : W_{(\lambda)} \to W_{(\lambda)}$ is an involution. We choose our basis of the semi-invariants of $W_{(\lambda)}$ to consist of eigenvectors (with eigenvalues ± 1) of ρ.

2.3. **Finding the vectors satisfying the h-condition.** At this point we have computed fixed bases of W and of the semi-invariants in W, and these bases respect as well as possible both the weight space decomposition of W and the self-duality map ρ. Now we come to computing the kernel of $1+h+h^2$ on the semi-invariant space.

Let M be the matrix representing $1+h+h^2$ with respect to these chosen bases, so that the columns of M are indexed by our basis of the space of semi-invariants in W and the rows of M are indexed by our basis of W. Thus if $\{s_j\}$ is our basis of the semi-invariant vectors, the j^{th} column of M contains the coefficients of $s_j(1+h+h^2)$ with respect to our chosen basis of W. We simply need to find the kernel of M. As we've mentioned, the difficulties arise from the very large size of the matrices M that arise. In particular for W_{120} we are faced with a 1771561 by 73800 matrix. This leads to problems with the time to compute the matrix M itself, with storing it, and with the time to row reduce it. Our solution is to compute only parts of M at any given time, to row reduce as we go, and to use the hard disk for storage. Fortunately the matrix is rather sparse, but it becomes less so as the row reduction progresses. To minimize this fill, we make use of our knowledge of the structure of the matrix.

Lemma 2.4. *If $w \in W$ is a weight vector of weight (a,b,c) then wh lies in the sum over all u and v of the weight spaces $W_{(u,v,c)}$.*

Proof. This is apparent from viewing W inside $V_g \otimes V_g^*$. □

We'll denote by W_c the sum of the $W_{(u,v,c)}$ and by $W_{(c)}$ the sum of $W_c\sigma$ for $\sigma \in P$. So $W_{(c)}$ is the sum of the weight spaces for those weights one of whose components is c.

Lemma 2.5. *If $c \not\equiv g \pmod 2$ then $W_{(c)}$ contains no nonzero semi-invariant vectors.*

Proof. If $W_{(c)}$ contains a nonzero vector invariant by S then, since S preserves the weight spaces of W, there must be a weight λ which has c as a component and for which W_λ contains a nonzero vector invariant by S. Thus by Lemma 2.3 c must be congruent to g modulo 2. □

For each fixed c we let M_c be the matrix whose rows are those rows of M that correspond to our chosen basis vectors of W_c. By Lemma 2.4 these rows will only have non-zero entries in the columns corresponding to our basis vectors of the semi-invariants in $W_{(c)}$. In particular, M_c is zero unless $c \equiv g \pmod 2$, by Lemma 2.5. For $c \equiv g \pmod 2$ we can efficiently compute M_c since we only need to compute the entries in those relatively few columns that do correspond to $W_{(c)}$. It is also comparatively easy to store a given M_c in memory, since we again only have to keep track of these relatively few columns.

We claim that the rows of M_c are highly dependent.

Proposition 2.6. *The rank of M_c is no more than the dimension of the h-invariants in W_c.*

Proof. Consider $(1+h+h^2)$ as a linear map

$$T : W \to (\sum_{z\neq c} W_z)\backslash W \cong W_c.$$

Then the rank of M_c is no more than the rank of T. But the image of T is h-invariant as $T \circ h = T$. □

For a random representation V of $\langle h \rangle$, one expects V^h to have dimension approximately $\frac{1}{3}$ the dimension of V. Indeed, this is what we observe with the spaces W_c.

Furthermore, we know which pairs M_c and M_d will share columns with nonzero entries. If (a, b, c) is any weight of W then $a + b + c = 3g$ since W has central character $\det^{3g}$. Moreover, each of a, b, c is no more than $2g$ since W has highest weight $(2g, g, 0)$. Thus the sum of any two of a, b, c must be at least g. In particular, if $c \neq d$ and $c+d < g$ then $M_{(c)} \cap M_{(d)} = \{0\}$ since no weight λ can simultaneously have c and d as entries.

Our algorithm for computing the kernel of M then begins by computing M_c for each $c \equiv g \pmod 2$, $c < g/2$. We perform Gaussian elimination on each of these M_c separately, putting each into reduced row-echelon form, and store the resulting non-zero rows to disk. We then continue with $g/2 \leq c \leq g$. Again we row reduce each M_c separately, but we must also perform further row reductions on M_c to remove any dependencies with the rows of any M_d with $g - c \leq d < c$. This still only requires holding two of the M_i in memory at any given time, if we are willing to spend some extra time reading and writing to disk. Once we have done this, we put the resulting M_c into reduced row-echelon form and store the non-zero rows to disk.

We thus obtain row-echelon forms $M_{\leq c}$ for initial segments of the matrix M by taking the concatenation of the reduced row-echelon forms of the M_e with $e \leq c$. Note that $M_{\leq c}$ is not in *reduced* row-echelon form since we do not adjust the rows of M_d in the step described above. We have found that to do so significantly increases the rate of fill in the matrix, and unacceptably slows down the calculations. However, we find that as c gets close to g, the rank of $M_{\leq c}$ gets fairly near to the number of columns of M, and $M_{\leq c}$ becomes far less sparse than the original matrix M. For example, when $g = 110$ we find that the rank of $M_{\leq 108}$ is 54603, while M has 56971 columns. This gives a co-rank of 2368, while the average number of nonzero entries in a row of $M_{\leq 108}$ is greater than 1400. At this point we computed the reduced row-echelon form of $M_{\leq 108}$, each of whose rows necessarily has no more nonzero entries than $1 + \operatorname{corank}(M_{\leq 108}) = 2369$. In fact, the average number of nonzero entries in each row of the reduced row-echelon form for $M_{\leq 108}$ was 228. In practice we put $M_{\leq c}$ into reduced row-echelon form once the average number of nonzero entries in a row exceeded twice the corank.

Finally, to compute remaining rows $(c > g)$ of M we make use of:

Lemma 2.7. *Suppose $w \in W_g$ is semi-invariant and that*

$$w(1 + h + h^2) = \sum_{\lambda} w_\lambda,$$

a sum of weight vectors. Let $\tau = \mathit{diag}(1, -1, -1)$. Then

$$\rho(w)(1 + h + h^2) = \sum_{\lambda} \lambda(\tau)\rho(w_\lambda),$$

again a sum of weight vectors.

Proof. Applying ρ we see $\rho(w(1 + h + h^2)) = \sum_\lambda \rho(w_\lambda)$. For any $u \in \mathrm{GL}(3, \mathbb{F})$ let $u^* = {}^t u^{-1}$. The we have

$$\rho(w(1 + h + h^2)) = \rho(w)(1 + h^* + (h^2)^*)$$

since $\det(h) = 1$. Note that $\tau h \tau = (h^2)^*$ and $\tau h^2 \tau = h^*$. Thus

$$\rho(w(1 + h + h^2)) = \rho(w)\tau(1 + h + h^2)\tau = \rho(w\tau^*)(1 + h + h^2)\tau.$$

Since w is semi-invariant, $w\tau^* = w$. So

$$\rho(w)(1+h+h^2)\tau = \sum_\lambda \rho(w_\lambda),$$

and

$$\rho(w)(1+h+h^2) = \sum_\lambda \rho(w_\lambda)\tau = \sum_\lambda \rho(w_\lambda \tau^*) = \sum_\lambda \lambda(\tau)\rho(w_\lambda)$$

as each w_λ is a weight vector of weight λ and $\tau^* = \tau$. □

Since we have chosen our basis of the semi-invariants to be stable under ρ, we can interpret the lemma as relating two columns of the matrix M for $(1+h+h^2)$. Alternatively, we can view it as relating the rows of $M_{(c)}$ to the rows of $M_{(2g-c)}$. Recall that the elements of our basis of the semi-invariants in W consists of pairs of vectors that are interchanged by ρ, along with eigenvectors of ρ with eigenvalue ± 1. Define a transformation T on the rows of M that exchanges the entries in columns corresponding to pairs $w \neq \rho(w)$, that negates the entries in columns corresponding to eigenvectors of ρ with eigenvalue -1 and that leaves alone the entries in columns corresponding to eigenvectors of ρ of eigenvalue 1. Then since ρ interchanges $W_{(c)}$ and $W_{(2g-c)}$, it follows from Lemma 2.3 that the span of the rows of M_{2g-c} is equal to the span of the rows of $T(M_c)$. Thus we can use the rows of $M_{\leq g-2}$ that we have computed above to generate rows whose span is equal to the span of the rows of M corresponding to $c > g$.

This allows us to do fewer calculations of the action of h on elements of W, and more importantly to do fewer row operations, since we have already removed redundant rows from $M_{\leq g-2}$. Further, the row operations we do have to do are not very time consuming since once we've put $M_{\leq g-2c}$ into reduced row-echelon form we've seen that it is not very dense. Indeed, the entire process for $c > g$ takes only about 5% of the time of the whole calculation.

As a further indication of the size of these calculations, we note that to compute the homology of W_{110} required slightly over half a billion row operations and took five and a half days.

2.4. **Results and checks.** We carried out this calculation for W_g with $0 \leq g \leq 120$. We used the prime $r = 223$ as the characteristic of our coefficient module when working with $g \leq 111$ and once $g \geq 112 = \lceil \frac{223}{2} \rceil$ we switched to $r = 281$. Using the theorems at the start of this section, we obtained:

Theorem 2.8 (Computational Result)**.** $\mathrm{H}^3_{cusp}(\Gamma, W_g(\mathbb{C})) = S(W_g(\mathbb{C}))$ *for* $0 \leq g \leq 120$.

It is worth remarking on a few checks on these calculations. We first note that prior calculations of the cohomology of W_g for small values of g were carried out by Eric Conrad and Tyler Neylon and were entirely independent of our programs. Their data and ours agree in all cases.

For even $g \leq 120$, we see $\dim \mathrm{H}^3(\mathrm{SL}_3(\mathbb{Z}), W_g(\mathbb{Z}/r)) = 3s_{g+2}(1) + 1$. The fact that we recovered the dimensions of the spaces of classical holomorphic forms is, in itself, a good check that we have obtained meaningful data. For small g we also computed the values of some of the Hecke operators (see [2] for a discussion of how this is done) and confirmed that they had the expected eigenvalues.

For odd g, the situation is less satisfying. Here we see that the cohomology in characteristic r is 0 dimensional. It is not terribly reassuring to have a program return "0" after running for a week! Fortunately, while the cohomology is generically 0 for odd r, there are particular pairs (r, g) for which we expect cohomology. In particular, in [2] we conjecture the existence of certain cohomology classes which correspond to Galois representations. Let $G_{\mathbb{Q}}$ be the absolute Galois group of $\mathbb{Q}$ and for a prime p let $I_p \subset G_{\mathbb{Q}}$ be an inertia subgroup at p, defined up to conjugacy. If p is congruent to 1 modulo 3 and $\rho : G_{\mathbb{Q}} \to \mathrm{SL}_3(\bar{\mathbb{F}}_p)$ is continuous, irreducible, unramified away from p, and has $\rho(I_p)$ noncentral of size 3 then we predict a cohomology class in $\mathrm{H}^3(\mathrm{SL}_3(\mathbb{Z}), W_{\frac{p-4}{3}}(\bar{\mathbb{F}}_p))$ which corresponds to ρ. The polynomial $x^4 - x^3 - 7x^2 + 2x + 9$ generates an A_4 extension K of $\mathbb{Q}$ ramified only at 163, with $\rho|_{I_{163}}$ as required. If we let $\rho : G_{\mathbb{Q}} \to Gal(K/\mathbb{Q}) \to \mathrm{GL}_3(\bar{\mathbb{F}}_p)$ then ρ has the properties described above and so we predict a nontrivial cohomology class in $\mathrm{H}^3(\mathrm{SL}_3(\mathbb{Z}), W_{53}(\mathbb{F}_{163}))$. We ran our programs in this case, and indeed the cohomology is 1 dimensional. The fact that our program detected this class for $h = 53$ and $p = 163$ and not for other values of p with $h = 53$ is a good check that they are computing what they are supposed to. (It is also a further confirmation of the conjecture in [2].)

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