RIGIDITY OF P-ADIC COHOMOLOGY CLASSES OF CONGRUENCE SUBGROUPS OF $GL(N, \mathbb{Z})$

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The main result of [Ash-Stevens 97] describes a framework for constructing p-adic analytic families of p-ordinary arithmetic Hecke eigenclasses in the cohomology of congruence subgroups of $GL(n)/\mathbb{Q}$ where the Hecke eigenvalues vary p-adic analytically as functions of the weight. Unanswered in that paper was the question of whether or not *every* p-ordinary arithmetic eigenclass can be "deformed" in a "positive dimensional family" of arithmetic eigenclasses. In this paper we make precise the notions of "deformation" and "rigidity" and investigate their properties. Rigidity corresponds to the non-existence of positive dimensional deformations other than those coming from twisting by the powers of the determinant. Formal definitions are given in §3. When $n = 3$, we will give a necessary and sufficient condition for a given p-ordinary arithmetic eigenclass to be p-adically rigid and we will use this criterion to give examples of p-adically rigid eigenclasses on $GL(3)$.

On the other hand, an argument of Hida in section 5 of [Hida 94] can be modified to prove the existence of nontrivial, non-p-torsion p-adic deformations (modulo twisting) for the same examples. Details will appear elsewhere. Here we merely note that the use of quasicuspidality (see Definition 6.1) seems to be essential to make Hida's arguments work in the $GL(3)$ case.

Coupled with our rigidity result (Theorem 9.4) this proves the existence of non-arithmetic p-adic deformations of p-ordinary arithmetic (cohomological) cuspforms for $GL(3)$. In fact, both of our examples are deformations of Hecke eigenclasses in the cohomology of congruence subgroups of $SL(3, \mathbb{Z})$ with trivial coefficients.

More precisely, we define a "Hecke eigenpacket" to be a map from the Hecke algebra to a coefficient ring which gives the Hecke eigenvalues attached to a Hecke eigenclass in the cohomology. We then consider p-adic analytic "deformations" of the Hecke eigenpacket. There are several advantages to this point of view, as opposed to deforming the cohomology classes themselves. For one thing, a surjective map of coefficient modules need not induce a surjective map on the cohomology,

but the long exact sequence in cohomology enables us to keep track of Hecke eigen-packets. For example, Lemma 4.3 is a surjectivity result on the level of Hecke eigenpackets proved in this way (see Theorem 5.1). Standard tools of algebra and geometry can be used to study the Hecke algebra and the deformations of the eigen-packets. Moreover, it is the Hecke eigenvalues that carry the interesting arithmetic information attached to a cohomological eigenclass.

Our main theoretical tool is the cohomology of congruence subgroups $\Gamma \subseteq GL(n, \mathbb{Z})$ with coefficients in the space of bounded measures on the big cell of $GL(n)$ (see §2). In §4 we define the ring R_{ord} as a quotient of the Hecke algebra acting on the p-ordinary part of this cohomology. In Theorem 4.2 we prove that R_{ord} is *universal* in the sense that every p-ordinary arithmetic Hecke eigenpacket factors through R_{ord}. We show that R_{ord} is a complete semilocal noetherian ring and study the geometry of its associated p-adic rigid analytic space $\mathcal{X}_{ord}$. This leads us to a necessary and sufficient condition for p-adic rigidity (Theorem 4.9), valid for any n. In sections 6 through 8 we recall a number of representation theoretic facts, which in case $n = 3$ allow us to prove a criterion for rigidity (Corollary 8.5) that is accessible to numerical computation. Our particular computations are described in §9.

The first and third authors have also developed a theory of p-adic deformations for non-ordinary (but finite slope) Hecke eigenpackets. Using a more representation-theoretic approach, [Emerton 04] has defined an eigenvariety in this context, gen-eralizing to $GL(n)$ the Coleman-Mazur eigencurve for $GL(2)$. Emerton states his results for any $\mathbb{Q}_p$-split reductive algebraic group and it is perhaps worth noting that our methods also apply *mutatis mutandis* in this more general context.

The question of p-adic rigidity arises again in the finite slope context. However, it will probably be impossible to verify by computations like those in this paper whether or not any particular non-ordinary eigenpacket is p-adically rigid. The reason is that in the non-ordinary case, we lose control over the "size" of the subset of the parameter ("weight") space over which the family exists. In the ordinary case, we know that the families exist over the whole parameter space, so we can test p-adic rigidity by looking at weights that are p-adically far from the given weight. Since the only weights for which computer calculations are currently feasible are p-adically rather far from each other, this limits our computational methods to the ordinary case.

One knows that the symmetric square lift to $GL(3)$ of any p-ordinary classi-cal newform f does indeed deform into a p-adic analytic family (lifting Hida's p-ordinary deformation of f). From [Ash-Stevens 86] we know that the cohomology classes associated to the symmetric square lifts are essentially self-dual eigenclasses (Definition 8.1) for $GL(3)$. In looking for examples of p-adically rigid eigenclasses we therefore examined the non-essentially-self-dual p-ordinary eigenclasses constructed numerically in [Ash-Grayson-Green 84]. The numerical calculations described in §9 show that these eigenclasses satisfy our criterion and are therefore p-adically rigid.

The calculations suggest that, outside of symmetric square lifts, non-torsion arithmetic cohomology is very sporadic. On the other hand, failure of rigidity of any non-symmetric square lift would give rise to an abundance of non-essentially-selfdual eigenclasses, at least if the Hecke eigenpacket is not essentially-selfdual modulo p. So we are led to conjecture:

Conjecture 0.1. *If a finite slope cuspidal Hecke eigenclass in the cohomology of*

a congruence subgroup of $GL(3, \mathbb{Z})$ is not essentially selfdual then it is p-adically rigid.

Recent work of Eric Urban suggests that the converse may be true for all $GL(n)$, $n \geq 2$. When $n = 2$, every cuspidal eigenclass is essentially selfdual, and in this case Hida (in the ordinary case) and Coleman (in general) proved that all finite slope cuspidal eigenclasses can be deformed into non-trivial p-adic analytic families, i.e they are not p-adically rigid in our sense.

In order to prove p-adic rigidity in the cases we study in the last section of the paper, we have to compute the cuspidal cohomology of $\Gamma_0(q)$ with small-dimensional irreducible coefficient modules when $n = 3$. Here, $\Gamma_0(q)$ denotes the arithmetic subgroup of $SL(n, \mathbb{Z})$ consisting of matrices whose top row is congruent to $(*, 0, \ldots, 0)$ modulo q, where q is a prime different from p.

Here are the computational results. Let V_h denote the irreducible finite-dimensional $GL(3, \mathbb{C})$-module of highest weight $(2h, h, 0)$. V_h is a submodule of $Sym^h(A) \otimes Sym^h(A^\dagger) \otimes det^h$ where A is the standard 3-dimensional complex representation of $GL(3, \mathbb{C})$ and $A^\dagger$ is its contragredient. Let E be a rational complex representation of $GL(3, \mathbb{C})$. As explained in section 8 below, the cuspidal cohomology of any congruence subgroup of $SL(3, \mathbb{Z})$ with coefficients in E is known to vanish if E contains no submodule isomorphic to a twist by a power of the determinant of any V_h. So to look for cuspidal classes, it suffices to consider cohomology with V_h coefficients.

Computational Results 0.1.

There is no cuspidal cohomology for $\Gamma_0(q)$ with coefficients in V_h for $h = 1, 2$, q any prime up to and including 223, and for $h = 3, 4$, q any prime up to and including 97.

The computations use the programs and methods outlined in section 8 of [Ash-Doud-Pollack 02]. Cuspidal cohomology can only occur in H^2 and H^3 and Poincaré duality shows that $H^2_{cusp} \simeq H^3_{cusp}$. So it suffices to compute H^3.

It is somewhat surprising that we found no non-essentially-selfdual cuspidal cohomology in these ranges, since for $h = 0$ (i.e. trivial coefficients) there exists non-essentially-selfdual cuspidal cohomology for $q = 53, 61, 79, 89$ [Ash-Grayson-Green 84].

As a consequence of these computations, also in section 9, we deduce the 3-adic rigidity of the cuspidal cohomology with trivial coefficients of level 89 and the 5-adic rigidity of the cuspidal cohomology of level 61. The difficulty of testing other examples is purely owing to our inability to compute for large h.

In this paper we avoid speaking about an eigenvariety for $GL(n)$, preferring instead to formulate our results in terms of eigenpackets. We look only at "arithmetic eigenpackets", i.e. those whose associated rigid analytic spaces are the Zariski closure of their arithmetic points (see definitions 3.1 and 3.6). In particular we say an arithmetic point is "rigid" if it is isolated on any *arithmetic* eigenpacket that contains it (see definition 3.9).

It may help to describe our picture roughly in terms of a putative eigenvariety. We focus on the $n = 3$ situation and let Y be the eigenvariety for $GL(3)$ modulo twisting. Y projects to a 2-dimensional weight space $\mathcal{W}$. If a point x on Y is a symmetric square lift we expect it will lie on a component of Y whose projection to $\mathcal{W}$ contains the 1-dimensional self-dual line. If x is not a symmetric square lift, then

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our conjecture above implies that any component of Y containing x will project down to a set in $\mathcal{W}$ that is either 1-dimensional and transverse to the self-dual line or 0-dimensional.

The contents of this paper are as follows. In Sections 1 and 2 we introduce the Hecke algebras and the cohomology groups of interest to us and describe the action of the Hecke algebra on the cohomology. Section 2 also recalls the main results from [Ash-Stevens 97] that form the basis of the rest of this paper. In section 3 we develop a formalism of Hecke eigenpackets and define the key notions of p-adic deformation and p-adic rigidity. In section 4, we introduce the p-ordinary arithmetic eigenvariety and in section 5 we prove its universality.

In sections 6 through 8 we collect the representation theoretic facts we need in order to give a testable criterion for p-adic rigidity. We then give our computational results in Section 9. In particular, we give two examples of p-adically rigid eigenclasses, one with $p = 3$ and level 89, the other with $p = 5$ and level 61.

Section 1: Hecke Operators and (T, S)-Modules

Let $G = GL(n)$ ($n \geq 2$) and fix a positive prime p. Let $T \subseteq G$ be the group of diagonal matrices and let $S \subseteq G(\mathbb{Q})$ be a subsemigroup containing a given congruence subgroup $\Gamma \subseteq SL_n(\mathbb{Z})$.

Definition 1.1. A (T, S)-module is a topological $\mathbb{Z}_p$-module V endowed with a continuous left action of $T(\mathbb{Q}_p)$ and a right action of S that commutes with the $T(\mathbb{Q}_p)$-action.

In the examples, V can always be viewed as a "highest weight module" in which the action of T is defined via multiplication by the highest weight character of $T(\mathbb{Q}_p)$.

Let Λ and Λ_T be the completed group rings on the topological groups $T(\mathbb{Z}_p)$ and $T(\mathbb{Q}_p)$ respectively. Then Λ is the Iwasawa algebra of $T(\mathbb{Z}_p)$ and, letting $T_0 \subseteq T(\mathbb{Q}_p)$ be the subgroup of matrices whose diagonal entries are integral powers of p, we have $\Lambda_T = \Lambda[T_0]$. The group T_0 is free abelian with generators $\pi_k :=$ diag$(1, \ldots, 1, p, \ldots, p)$ (k copies of p), for $k = 1, \ldots, n$, and consequently we have

$$\Lambda_T = \Lambda[\pi_1, \pi_1^{-1}, \cdots, \pi_n, \pi_n^{-1}],$$

a Laurent polynomial ring in n variables over Λ. The element

$$\pi := \pi_1 \cdots \pi_{n-1}$$

plays a special role in our theory.

Since $T(\mathbb{Q}_p)$ acts continuously on any (T, S)-module V, we may regard V as a right $\Lambda_T[S]$-module. Indeed, this defines an equivalence of categories between the category of (T, S)-modules and the category of continuous $\Lambda_T[S]$-modules.

Now fix a positive integer N with $p \nmid N$ and define the congruence subgroups

$$\Gamma := \{g \in SL(n, \mathbb{Z}) \mid \text{top row of } g \equiv (*, 0, \ldots, 0) \bmod N\}, \text{and}$$

$$\Gamma_0 := \{g \in \Gamma \mid g \text{ is upper triangular modulo } p \}.$$

We define the subsemigroups S' and S'_0 of $M_n^+(\mathbb{Z})$ (the plus denotes positive determinant) by

$$S' := \{s \in M_n^+(\mathbb{Z}) \mid (\det(s), Np) = 1, \text{ top row of } s \equiv (*, 0, \ldots, 0) \bmod N\}$$

$$S'_0 := \{s \in S' \mid s \text{ is upper triangular modulo } p\}.$$

Finally, we define the subsemigroups S and S_0 of $G(\mathbb{Q}_p)$:

$$S := \text{semigroup generated by } S' \text{ and } \pi,$$
$$S_0 := \text{semigroup generated by } S_0' \text{ and } \pi.$$

For $s \in S$ we let $T(s)$ be the double coset $\Gamma s \Gamma$. It is well known that the double coset algebra $D_{\Lambda_T}(\Gamma, S) = \Lambda_T[\Gamma \backslash S/\Gamma]$ is the free polynomial algebra over Λ_T generated by $T(\pi)$ and the elements

$$T(\ell, k) := T(s), \ s = \text{diag}\,(1, \ldots, 1, \ell, \ldots, \ell), \text{ with } k \text{ copies of } \ell\text{'s } (1 \leq k \leq n),$$

for all primes l not dividing Np. (See Props 3.16, 3.30 and Thm 3.20 of [Shimura 71].)

The double coset algebra $D_{\Lambda_T}(\Gamma_0, S_0)$ is likewise a free polynomial algebra generated by the Γ_0-double cosets of π and of the diag $(1, \ldots, 1, \ell, \ldots, \ell)$ with l a prime not dividing Np and $1 \leq k \leq n$. Thus the map sending a polynomial expression in the generators of $D_{\Lambda_T}(\Gamma, S)$ to the corresponding polynomial expression in the generators of $D_{\Lambda_T}(\Gamma_0, S_0)$ gives an isomorphism $D_{\Lambda_T}(\Gamma, S) \longrightarrow D_{\Lambda_T}(\Gamma_0, S_0)$.

We define the element U to be $T(\pi)$ and let $\mathcal{H}$ denote the Λ-subalgebra of $D_{\Lambda_T}(\Gamma, S)$ generated by the elements U and the $T(\ell, k)$:

$$\mathcal{H} = \Lambda \left[U, \ T(\ell, k) \ \middle| \ \ell \nmid Np \text{ is prime and } 1 \leq k \leq n \right].$$

We let $\mathcal{H}_0$ be the image of $\mathcal{H}$ in $D_{\Lambda_T}(\Gamma_0, S_0)$ and note that

$$\mathcal{H} \xrightarrow{\sim} \mathcal{H}_0$$

is an isomorphism of Λ-modules.

We define a modified action of S, which we call the $\star$-action, as follows. There is a unique multiplicative map $\tau : S \longrightarrow T_0$ that is trivial on S' and sends π to itself. For any (T, S)-module V we define the $\star$-action of S on V by the formula

$$w \star \sigma := \tau(\sigma)^{-1} \cdot w\sigma.$$

for $w \in V$ and $\sigma \in S$. We then define $V^\star$ to be V endowed with the $\star$-action of S on the right and the given action of Λ on the left. For any subquotient L of $V^\star$ the double coset algebra $D_\Lambda(\Gamma, S)$ acts naturally on the Γ-cohomology of L. We then let $\mathcal{H}$ act on the cohomology of L.

We apply this construction verbatim for any (T, S_0)-module V and similarly obtain a $\Lambda[S_0]$-module $V^\star$. Then for any $\Lambda[S_0]$-subquotient L of $V^\star$ the Γ_0-cohomology of L receives a natural $\mathcal{H}_0$-module structure. Using our fixed isomorphism $\mathcal{H} \cong \mathcal{H}_0$, we may (and do) regard the Γ_0-cohomology of L as an $\mathcal{H}$-module. For example, if V is a (T, S)-module and L is a subquotient of $V^\star$ then we may also regard V as a (T, S_0)-module and L as an $\Lambda[S_0]$-subquotient of $V^\star$. So both $H^*(\Gamma, L)$ and $H^*(\Gamma_0, L)$ are endowed with natural $\mathcal{H}$-module structures. Note, however, that in general the restriction map

$$H^*(\Gamma, L) \longrightarrow H^*(\Gamma_0, L)$$

does *not* commute with U, though it does commute with the Hecke operators $T(\ell, k)$ for $\ell \nmid Np$.

Note that we use throughout the notation H^* to stand for $\oplus_k H^k$.

Let B be the Borel subgroup of upper triangular matrices and denote the group of lower triangular matrices by B^{opp}. Let λ denote the highest weight, with respect to (B^{opp}, T), of a finite dimensional irreducible right rational $\mathbb{Q}_p[G]$-module V_λ. Then V_λ inherits a natural action of S on the right and we let $T(\mathbb{Q}_p)$ act (continuously) on the left via the character λ. In a moment, we will also choose a $\mathbb{Z}_p$-lattice $L_\lambda \subseteq V_\lambda$ that is invariant under S'_0 and satisfies $\pi^{-1}L_\lambda\pi \subseteq L_\lambda$. Thus by the discussion of the last section, we obtain an action of $\mathcal{H}$ on $H^*(\Gamma, V_\lambda)$, $H^*(\Gamma_0, V_\lambda)$, and $H^*(\Gamma_0, L_\lambda)$.

Now let N^{opp} be the unipotent radical of B^{opp} and let $Y := N^{opp}(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)$. We also let

$$X := \text{the image of } G(\mathbb{Z}_p) \text{ in } Y, \text{ and}$$

$$X_0 := \text{the image of } I \text{ in } Y,$$

where I is the Iwahori subgroup of $G(\mathbb{Z}_p)$ consisting of matrices that are upper triangular modulo p. The semigroup S acts on Y by multiplication on the right and $T(\mathbb{Q}_p)$ acts by multiplication on the left. Both X and X_0 are stable under $T(\mathbb{Z}_p)$. Also X is stable under the action of S' and X_0 is stable under S'_0. We define a new action of S on Y by setting for $\sigma \in S$

$$N^{opp}(\mathbb{Q}_p)g \star \sigma = N^{opp}(\mathbb{Q}_p)\tau(\sigma)^{-1}g\sigma,$$

where $\tau\colon S \to T_0$ is the homomorphism sending π to π and acting trivially on S' introduced above. This new $\star$-action is well defined since π normalizes $N^{opp}(\mathbb{Q}_p)$. Moreover, X_0 is stable under the $\star$-action of S_0 since every element of X_0 is represented by some element $b \in B(\mathbb{Z}_p)$ and $\pi^{-1}B(\mathbb{Z}_p)\pi \subset B(\mathbb{Z}_p)$. We will henceforth use this $\star$-action when we act S on Y or S_0 on X_0.

For any open subset $Z \subseteq Y$, we let $\mathbb{D}(Z)$ be the space of $\mathbb{Z}_p$-valued measures on Z, so $\mathbb{D}(Z) = \text{Hom}_{\mathbb{Z}_p}(C(Z), \mathbb{Z}_p)$, where $C(Z)$ is the space of all compactly supported continuous $\mathbb{Z}_p$-valued functions on Z. By functoriality (using the $\star$-action), $\mathbb{D}(Y)$ inherits a natural (T, S)-structure. By extending measures by zero, we obtain a $\Lambda[S_0]$-embedding $\mathbb{D}(X_0) \hookrightarrow \mathbb{D}(Y)$. In [Ash-Stevens 97], the space $\mathbb{D} = \mathbb{D}(X)$ is given a $\Lambda[S]$ structure by setting for $\mu \in \mathbb{D}$ and $f \in C(X)$

$$(\mu \star s)(f) = \mu((s \star \tilde{f})|_X)$$

where $\tilde{f}$ is the extension by zero of f to Y and $(s \star \tilde{f})(y) = \tilde{f}(y \star s)$. Hence the cohomology groups $H^*(\Gamma_0, \mathbb{D}(X_0))$, $H^*(\Gamma, \mathbb{D})$ are endowed with natural $\mathcal{H}$-module structures. Moreover, restriction of distributions from X to X_0 induces a $\Lambda[S_0]$-morphism $\mathbb{D} \longrightarrow \mathbb{D}(X_0)$ and thus induces a map

$$H^*(\Gamma, \mathbb{D}) \longrightarrow H^*(\Gamma_0, \mathbb{D}(X_0))$$

that commutes with action of $\mathcal{H}$. (To see that this maps commutes with $T(\pi)$ requires an explicit calculation similar to the proof of Proposition 4.2 of [Ash-Stevens 97].)

For any dominant integral weight λ, we define the map

$$\phi_\lambda\colon \mathbb{D}(Y) \longrightarrow V_\lambda$$

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by $\phi_\lambda(\mu) = \int_{X_0} v_\lambda x \, d\mu(x)$, where v_λ denotes a fixed highest weight vector in V_λ. One checks easily that ϕ_λ is a morphism of (T, S)-modules. Thus ϕ_λ induces a $\Lambda[S_0]$-morphism

$$\phi_\lambda : \mathbb{D}(X_0) \longrightarrow V_\lambda^\star.$$

We let L_λ be the image of this map and note that since $\mathbb{D}(X_0)$ is compact L_λ is a finitely generated I-invariant submodule of V_λ, hence is a lattice in V_λ (since V_λ is irreducible). This is the lattice whose existence was asserted in the first paragraph of this section. Thus ϕ_λ induces a natural $\mathcal{H}$-morphism

$$\phi_\lambda \colon H^*(\Gamma, \mathbb{D}) \to H^*(\Gamma_0, L_\lambda).$$

For any compact $\mathbb{Z}_p$-module H with a continuous action of the operator U on it, we define the ordinary part of H to be $H^0 = \cap U^i H$, where i runs over all positive integers. Then H^0 is the largest submodule of H on which U acts invertibly, and $H \mapsto H^0$ is an exact functor.

The following theorem is Theorem 5.1 of [Ash-Stevens 97].

Theorem 2.1. $H^*(\Gamma, \mathbb{D})^0$ *is finitely generated as a Λ-module. Moreover, for every dominant integral weight λ, the kernel of the map*

$$\phi_\lambda \colon H^*(\Gamma, \mathbb{D})^0 \to H^*(\Gamma_0, L_\lambda)^0$$

is $I_\lambda H^*(\Gamma, \mathbb{D})^0$ where I_λ is the kernel of the homomorphism $\lambda^\dagger : \Lambda \longrightarrow \mathbb{Z}_p$ induced by λ.

For easy reference, we will also cite here the two main lemmas from [Ash-Stevens 97] that we need to prove Theorem 2.1. We first need a few definitions. We let

$$W := \operatorname{Spf}(\Lambda)$$

be the formal scheme associated to Λ. We note that for each local field $K/\mathbb{Q}_p$ the K-points of W are in one-one correspondence with continuous homomorphisms $k : T(\mathbb{Z}_p) \longrightarrow K^\times$. If $t \in T(\mathbb{Z}_p)$ and $k \in W(K)$, then we will write $t^k \in K^\times$ for the value of k on t. Dominant integral weights can (and will) be viewed as elements of $W(\mathbb{Q}_p)$. We let W^+ denote the set of dominant integral weights.

Now let $k \in W(\mathbb{Q}_p)$. We let $C_k(X)$ denote the $\mathbb{Z}_p$-module

$$C_k(X) := \left\{ f : X \longrightarrow \mathbb{Z}_p \; \middle| \; \begin{array}{l} f \text{ is continuous, and} \\ f(tx) = t^k f(x) \text{ for all } t \in T(\mathbb{Z}_p), \; x \in X \end{array} \right\}.$$

Let $\mathbb{D}_k$ be the space of continuous $\mathbb{Z}_p$-valued linear functionals on $C_k(X)$. Clearly $C_k(X) \subseteq C(X)$ is a closed subspace, so restriction of linear functionals induces a surjective map $\mathbb{D} \longrightarrow \mathbb{D}_k$. We regard both $\mathbb{D}$ and $\mathbb{D}_k$ as $\Lambda[S]$-modules (with the $\star$-action of S). Let $I_k \subseteq \Lambda$ be the prime associated to $k \in W(\mathbb{Q}_p)$. The action of Λ on $\mathbb{D}_k$ factors through Λ/I_k.

Lemma 2.2. *The ideal I_k is generated by a $\mathbb{D}$-regular sequence. Moreover, the canonical sequence*

$$0 \longrightarrow I_k \mathbb{D} \longrightarrow \mathbb{D} \longrightarrow \mathbb{D}_k \longrightarrow 0$$

is exact.

Proof: Lemma 1.1 of [Ash-Stevens] shows that I_k is generated by a finite $\mathbb{D}$-regular sequence. We've already noted above that the map $\mathbb{D} \longrightarrow \mathbb{D}_k$ is surjective. We clearly have $I_k \mathbb{D}$ is contained in the kernel and it follows from Lemma 6.3 of [Ash-Stevens 97] that $I_k \mathbb{D}$ contains the kernel. Thus the sequence is exact. This completes the proof.

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Lemma 2.3. *If $\lambda \in W^+$ is a dominant integral weight, then the canonical map*

$$H^*(\Gamma, \mathbb{D}_\lambda)^0 \xrightarrow{\sim} H^*(\Gamma_0, L_\lambda)^0$$

is an $\mathcal{H}$-isomorphism.

Proof. The surjectivity of the map follows from Lemma 6.7 and the injectivity follows from Lemma 6.8 of [Ash-Stevens 97]. (Note, however, that the statement of Lemma 6.8 contains a typo. The conclusion of the statement should read: "Then ζ can be represented by a cocycle z_m in $\mathrm{Hom}_\Gamma(F_k, \mathbb{D})$ which takes values in $\mathbb{D}^*$.")

Remark 2.4. In an upcoming paper by the first and third authors, we generalize the ideas of this section to the non-ordinary "finite slope" situation.

SECTION 3: EIGENPACKETS: TWISTS AND DEFORMATIONS

Definition 3.0 If A is a Λ-module, a Λ-homomorphism $\varphi : \mathcal{H} \longrightarrow A$ will be called an A-valued eigenpacket. If $\varphi' : \mathcal{H} \longrightarrow A'$ is another eigenpacket then a morphism $f : \varphi \longrightarrow \varphi'$ is defined to be a Λ-morphism $f : A \longrightarrow A'$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\varphi} & A \\
& {\scriptstyle \varphi'} \searrow & \downarrow f \\
& & A'
\end{array}
$$

is commutative. Conversely, if $f : A \longrightarrow A'$ is a Λ-morphism, then we may push φ forward by f to obtain an eigenpacket $\varphi' := f_*\varphi$ and a morphism $f : \varphi \longrightarrow \varphi'$. If φ is an eigenpacket, we will denote its target by A_φ.

If A is either a $\mathbb{Q}_p$-Banach algebra or a $\mathbb{Z}_p$-algebra that is separated and complete in the p-adic topology, then there is a one-one correspondence between continuous homomorphisms $\Lambda \longrightarrow A$ and continuous characters $T(\mathbb{Z}_p) \longrightarrow A^\times$. This is also true for any $\mathbb{Q}_p$-Banach algebra A. For a character $\chi : T(\mathbb{Z}_p) \longrightarrow A^\times$ we let $\chi^\dagger : \Lambda \longrightarrow A$ denote the associated homomorphism. Conversely, if A is a Λ-algebra, we let χ_A denote the associated character and call χ_A the structure character of A. For any eigenpacket φ we define the *weight* of φ to be the structure character χ_{A_φ}, which we will denote simply χ_φ.

Let $K/\mathbb{Q}_p$ be a finite field extension. Whenever we say "K-valued eigenpacket" we will implicitly assume that K has been endowed with a Λ-algebra structure.

We are interested in eigenpackets that occur in the Γ_0-cohomology of finite dimensional irreducible rational representations of $GL(n)$. The weight of any such eigenpacket must be a dominant integral character of $T(\mathbb{Z}_p)$.

Definition 3.1 Let $K/\mathbb{Q}_p$ be a finite field extension and let φ be a K-valued eigenpacket. We say φ is *arithmetic of weight λ*, where λ is a dominant integral character, if φ occurs in $H^*(\Gamma_0, V_\lambda(K))$ (i.e. is the homomorphism of eigenvalues of some Hecke eigenvector).

Definition 3.2. Given an A-valued eigenpacket φ and a continuous character $\psi : \mathbb{Z}_p^\times \longrightarrow A^\times$ we may "twist" φ by ψ as follows. Let $A(\psi)$ be the unique continuous

Λ-algebra whose underlying $\mathbb{Z}_p$-algebra is A and whose structure character is $\chi_A \cdot (\psi \circ \det)$. The ψ-twist of φ is defined to be the unique eigenpacket

$$\mathrm{Tw}_\psi(\varphi) : \mathcal{H} \longrightarrow A(\psi)$$

given on Hecke operators by $U \longmapsto \varphi(U)$ and $T(\sigma) \longmapsto \psi(\det(\sigma))\varphi(T(\sigma))$ for $\sigma \in S'$. In particular, we define the *standard twist* of φ to be

$$\mathrm{Tw}(\varphi) := \mathrm{Tw}_{\mathbf{1}}(\varphi)$$

where $\mathbf{1} : \mathbb{Z}_p^\times \longrightarrow A^\times$ is the identity character. Iteration of the standard twist gives us the *n-fold twist* $\mathrm{Tw}^n(\varphi)$ for any integer n. We also define the *universal twist* of φ by

$$\widetilde{\mathrm{Tw}}(\varphi) := \mathrm{Tw}_{\tilde\psi}(i_*\varphi)$$

where $i : A \longrightarrow A[[\mathbb{Z}_p^\times]] := A \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ is the base extension morphism and $\tilde\psi : \mathbb{Z}_p^\times \longrightarrow A[[\mathbb{Z}_p^\times]]$ the universal character.

Note that by definition, the weight of $\mathrm{Tw}_\psi(\varphi)$ is the product of χ_φ and $\psi \circ \det$. Moreover, for any $\psi : \mathbb{Z}_p^\times \longrightarrow A^\times$ there is a unique A-morphism $\widetilde{\mathrm{Tw}}(\varphi) \longrightarrow \mathrm{Tw}_\psi(\varphi)$. The next proposition states that arithmeticity of eigenpackets is invariant under twist.

Proposition 3.3. *If φ is arithmetic, then $\mathrm{Tw}^n(\varphi)$ is arithmetic for every $n \in \mathbb{Z}$. If, moreover, φ has weight λ, then $\mathrm{Tw}^n(\varphi)$ has weight $\lambda \cdot \det^n$.*

Proof: Let λ be a dominant integral weight and let φ be an eigenpacket occurring in $H^*(\Gamma_0, V_\lambda(K))$. Now let $V_\lambda(n)$ be V_λ with the G-action twisted by the n-fold power of the determinant. Then the identity map $V_\lambda \longrightarrow V_\lambda(n)$ is a Γ_0-isomorphism and therefore induces an isomorphism $H^*(\Gamma_0, V_\lambda(K)) \longrightarrow H^*(\Gamma_0, V_\lambda(n)(K))$. A simple calculation shows that this map sends any φ-eigenvector to a $\mathrm{Tw}^n(\varphi)$-eigenvector. Thus $\mathrm{Tw}^n(\varphi)$ occurs in $H^*(\Gamma_0, V_\lambda(n)(K))$. It follows that $\mathrm{Tw}^n(\varphi)$ is arithmetic of weight $\lambda \cdot \det^n$. This completes the proof of Proposition 3.3.

Consider a topological Λ-algebra A which is of one of the following two types:
(1) a $\mathbb{Q}_p$-affinoid algebra; or
(2) a complete semi-local noetherian Λ-algebra with finite residual fields.

In either case, we assume the structure morphism $\Lambda \longrightarrow A$ is continuous.

Remark: Alternatively, we could work in the category of affine formal schemes over $\mathrm{Spf}(\Lambda)$, and indeed for all of our applications it suffices to work in that category. In that case, the ring of rigid analytic functions on an admissible affinoid open set is of type (1), while on an affine formal scheme finite over $\mathrm{Spf}(\Lambda)$, the ring of global rigid analytic functions of norm ≤ 1 is of type (2).

If A is arbitrary of type (1) or (2) we will say A is *p-analytic* and in this case an A-valued eigenpacket will be called *p*-analytic. In case (1) we will say φ is an *affinoid* eigenpacket. These conventions are justified by the fact that when A is *p*-analytic there is a naturally defined *p*-adic rigid analytic space $\mathcal{X}_A$ over $\mathbb{Q}_p$

such that for every finite extension $K/\mathbb{Q}_p$ the K-valued points of $\mathcal{X}_A$ are given by continuous homomorphisms from A to K:

$$\mathcal{X}_A(K) = \mathrm{Hom}_{cont}(A, K).$$

(For the construction of $\mathcal{X}_A$ in case (2) see section 7 of [de Jong 96], or section 1.1 of [Coleman-Mazur 98].) We will also write $\mathcal{X}_A(\overline{\mathbb{Q}}_p) := \mathrm{Hom}_{cont}(A, \overline{\mathbb{Q}}_p)$ for the set of $\overline{\mathbb{Q}}_p$-valued points and note that every $\overline{\mathbb{Q}}_p$-valued point must actually take values in some finite extension of $\mathbb{Q}_p$.

We give $\mathcal{X}_A(\overline{\mathbb{Q}}_p)$ the Zariski topology whose closed sets are the zero sets of the ideals of A. Note that if A is of type (1) this is the usual topology, but if A is of type (2) it is courser than the usual topology defined in terms of the full ring of rigid analytic functions.

Henceforth, all eigenpackets will be assumed to be p-analytic. Thus for every eigenpacket φ, the Λ-algebra A_φ is p-analytic. We denote the associated rigid analytic space by $\mathcal{X}_\varphi$. We will often use geometric language to describe properties of φ. For example, we say φ is affinoid if A_φ is an affinoid algebra and we say φ is irreducible if $\mathcal{X}_\varphi$ is an irreducible topological space.

Definition 3.4. A subset $\mathcal{S} \subseteq \mathcal{X}_A(\overline{\mathbb{Q}}_p)$ is said to be Zariski dense if 0 is the only element of A that vanishes at every point in $\mathcal{S}$.

We define the *weight space* $\mathcal{W}$ for $GL(n)$ by $\mathcal{W} := \mathcal{X}_\Lambda$. For any p-analytic Λ-algebra A the homomorphism $\Lambda \longrightarrow A$ induces a rigid analytic morphism

$$\mathrm{wt} : \mathcal{X}_A \longrightarrow \mathcal{W}$$

which we call the *weight map*. We let $\mathcal{W}^+$ be the set of all dominant integral weights and if A is a p-analytic Λ-algebra then for any field K finite over $\mathbb{Q}_p$ we define

$$\mathcal{X}_A^+(K) := \left\{ x \in \mathcal{X}_A(K) \;\middle|\; \mathrm{wt}\,(x) \in \mathcal{W}^+ \right\}.$$

We let $\mathcal{X}_A^+ := \mathcal{X}_A^+(\overline{\mathbb{Q}}_p)$ and call the points of $\mathcal{X}_A^+$ the dominant integral points on $\mathcal{X}_A$.

Remark 3.5. The set $\mathcal{W}^+$ of dominant integral points is a Zariski dense subset of $\mathcal{W}$. However the set $\mathcal{X}_A^+$ is not, in general, Zariski dense in $\mathcal{X}_A$.

If φ is an eigenpacket then we say φ is an eigenpacket over $\mathcal{X}_\varphi$. For each rigid analytic subspace $Z \subseteq \mathcal{X}_\varphi$, we let φ_Z be the composition of $\varphi : \mathcal{H} \longrightarrow A_\varphi$ with the map $A_\varphi \longrightarrow A(Z)$ defined by restriction to Z of rigid analytic functions on $\mathcal{X}_\varphi$. In this case φ_Z will be called the restriction of φ to Z. In particular, if $x \in \mathcal{X}_\varphi(\overline{\mathbb{Q}}_p)$, then the restriction of φ to $\{x\}$ will be called the specialization of φ at x and denoted φ_x. We define

$$\mathcal{A}(\varphi) := \left\{ x \in \mathcal{X}_\varphi(\overline{\mathbb{Q}}_p) \;\middle|\; \varphi_x \text{ is arithmetic} \right\}$$

and note that $\mathcal{A}(\varphi) \subseteq \mathcal{X}_\varphi^+$.

Definition 3.6. An eigenpacket φ is said to be
(1) *arithmetic* if the set $\mathcal{A}(\varphi)$ is Zariski dense in $\mathcal{X}_\varphi$, and
(2) $\overline{\mathbb{Q}}_p$-*valued* if the underlying ring of A_φ is a finite field extension of $\mathbb{Q}_p$.

If φ is an arithmetic eigenpacket, then we regard the family of all specializations φ_x, $x \in \mathcal{X}_\varphi(\overline{\mathbb{Q}}_p)$, as an analytic family of $\overline{\mathbb{Q}}_p$-valued eigenpackets that interpolates the arithmetic eigenpackets φ_x, $x \in \mathcal{A}(\varphi)$. We sometimes prefer to take a more *local* point of view, starting with a $\overline{\mathbb{Q}}_p$-eigenpacket and then "deforming" it to an analytic family. With this picture in mind, we make the following definition.

Definition 3.7. Given a K-valued eigenpacket φ_0, a *deformation* of φ_0 is a pair (φ, x) consisting of an eigenpacket φ and a point $x \in \mathcal{X}_\varphi(K)$ such that
(1) A_φ is a p-analytic integral domain; and
(2) $\varphi_x = \varphi_0$.
We say (φ, x) is an arithmetic deformation of φ_0 if, in addition to (1) and (2), we also have
(3) φ_Z is arithmetic for every admissible affinoid neighborhood Z of x in $\mathcal{X}_\varphi$.

Although we won't have use for it later, we record here a notion of equivalence of deformations. We say two deformations (φ, x), (φ', x') of φ_0 are locally equivalent if there exists an irreducible affinoid neighborhood Z of x in $\mathcal{X}_\varphi$, an irreducible affinoid neighborhood Z' of x' in $\mathcal{X}_{\varphi'}$, and an isomorphism $f : Z' \xrightarrow{\sim} Z$ such that $f(x') = x$ and $f^* \varphi_Z = \varphi'_{Z'}$.

Let $\tilde{\varphi}_0 = \widetilde{\mathrm{Tw}}(\varphi_0) : \mathcal{H} \longrightarrow \overline{\mathbb{Q}}_p[[\mathbb{Z}_p^\times]]$ be the universal twist of φ_0 and let $x_0 \in \mathcal{X}_{\tilde{\varphi}_0}(\overline{\mathbb{Q}}_p)$ be the point associated to the trivial character of $\mathbb{Z}_p^\times$. It follows from proposition 3.3 that if φ_0 is arithmetic, then the restriction of $\tilde{\varphi}_0$ to the connected component of x_0 is an arithmetic deformation of φ_0 at x_0.

Definition 3.8. An arbitrary deformation (φ, x) of φ_0 will be called a *twist deformation* of φ_0 if there is a morphism $f : \tilde{\varphi}_0 \longrightarrow \varphi$ such that $f^\#(x) = x_0$.

Note that if (φ, x) is a twist deformation of φ_0, then every specialization φ_z, $z \in \mathcal{X}_\varphi(\overline{\mathbb{Q}}_p)$, is a twist of φ_0 by some character $\mathbb{Z}_p^\times \longrightarrow \overline{\mathbb{Q}}_p^\times$.

Clearly, every twist deformation of an arithmetic $\overline{\mathbb{Q}}_p$-eigenpacket φ_0 is arithmetic. If the converse is true, we say φ_0 is arithmetically rigid.

Definition 3.9. An arithmetic $\overline{\mathbb{Q}}_p$-eigenpacket is said to be *arithmetically rigid* if every arithmetic deformation of φ_0 is a twist deformation.

In section 9 of this paper, we will give examples of arithmetic K-eigenpackets that are arithmetically rigid.

Section 4: The Universal p-Ordinary Arithmetic Eigenpacket

In this section we construct the universal p-ordinary arithmetic eigenpacket φ_{ord} and use it to give a criterion for rigidity of p-ordinary arithmetic $\overline{\mathbb{Q}}_p$-eigenpackets. Here K, as always, denotes a local field finite over $\mathbb{Q}_p$.

Definition 4.1. A $\overline{\mathbb{Q}}_p$-valued eigenpacket φ is said to be p-*ordinary* if $\varphi(U) \in \mathcal{O}_{\overline{\mathbb{Q}}_p}^\times$. More generally, a p-analytic eigenpacket φ is said to be p-ordinary if φ_x is p-ordinary for every $x \in \mathcal{X}_\varphi(\overline{\mathbb{Q}}_p)$.

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We have the following important result.

Theorem 4.2. *There is a p-ordinary arithmetic eigenpacket φ_{ord} satisfying the following universal property: for every p-ordinary arithmetic eigenpacket φ there is a unique morphism $\varphi_{ord} \longrightarrow \varphi$. Moreover, a point $x \in \mathcal{X}_{\varphi_{ord}}$ is arithmetic if and only if x has dominant integral weight. In other words: $\mathcal{A}(\varphi_{ord}) = \mathcal{X}^+_{\varphi_{ord}}$.*

Our construction of φ_{ord} will use the ordinary part of the cohomology of the universal highest weight module $\mathbb{D}$ constructed in §2. Consider the $\mathcal{H}$-module $\mathbb{H}$ defined by

$$\mathbb{H} := \bigoplus_i H^i(\Gamma, \mathbb{D})^0.$$

We define R'_{ord} to be the image of $\mathcal{H}$ in $\mathrm{End}_\Lambda(\mathbb{H})$ and consider the associated Λ-morphism

$$\varphi'_{ord} : \mathcal{H} \longrightarrow R'_{ord}.$$

We know from Theorem 2.1 that $\mathbb{H}$ is a finitely generated Λ-module, and from this it follows that R'_{ord} is a finite Λ-algebra. Thus φ'_{ord} is a p-analytic eigenpacket. Let $\mathcal{X}_{\varphi'_{ord}}$ be the associated rigid analytic space, and let $\mathcal{A}(\varphi'_{ord})$ be the set of arithmetic points on $\mathcal{X}_{\varphi'_{ord}}$ as defined in the last section.

Lemma 4.3. *Let φ be a p-ordinary $\overline{\mathbb{Q}}_p$-valued eigenpacket. Then φ is arithmetic if and only if $\varphi = (\varphi'_{ord})_x$ for some $x \in \mathcal{X}^+_{\varphi'_{ord}}$. In particular, $\mathcal{A}(\varphi'_{ord}) = \mathcal{X}^+_{\varphi'_{ord}}$.*

The proof of Lemma 4.3 will be given in the next section. In this section, we assume Lemma 4.3 and prove Theorem 4.2 as a consequence.

Proof of Theorem 4.2. Let $I_{arith} \subseteq R'_{ord}$ be the ideal defined by

$$I_{arith} := \bigcap_{x \in \mathcal{A}(\varphi'_{ord})} \ker(x).$$

We then let $R_{ord} := R'_{ord}/I_{arith}$ and define

$$\varphi_{ord} : \mathcal{H} \longrightarrow R_{ord}$$

to be the composition of φ'_{ord} with $R'_{ord} \longrightarrow R_{ord}$. By definition, we see that φ_{ord} is arithmetic. Let $J := \ker(\varphi_{ord}) \subseteq \mathcal{H}$.

Now let $\varphi : \mathcal{H} \longrightarrow A_\varphi$ be an arbitrary p-ordinary arithmetic eigenpacket. We will show $J \subseteq \ker(\varphi_x)$ for every $x \in \mathcal{A}(\varphi)$. Indeed, if $x \in \mathcal{A}(\varphi)$ then, since φ_x is arithmetic, Lemma 4.3 tells us that there is a point $y \in \mathcal{A}(\varphi'_{ord}) = \mathcal{X}^+_{\varphi'_{ord}}$ such that $\varphi_x = (\varphi'_{ord})_y$. But we clearly have $J \subseteq \ker((\varphi'_{ord})_y)$, so $J \subseteq \ker(\varphi_x)$, as claimed.

It now follows easily that $J \subseteq \ker(\varphi)$. Indeed, from the last paragraph we have

$$\varphi(J) \subseteq \bigcap_{x \in \mathcal{A}(\varphi)} \ker(x),$$

and the intersection on the right is $\{0\}$ since φ is arithmetic. Thus we have shown that $\ker(\varphi_{ord}) \subseteq \ker(\varphi)$. Since $\varphi_{ord} : \mathcal{H} \longrightarrow R_{ord}$ is surjective by definition, we conclude that $\varphi : \mathcal{H} \longrightarrow A_\varphi$ factors through φ_{ord}. This proves the existence of

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a morphism $\varphi_{ord} \longrightarrow \varphi$. Uniqueness follows from the surjectivity of φ_{ord}. This completes the proof of Theorem 4.2.

It is natural to ask whether or not the morphism $\varphi'_{ord} \longrightarrow \varphi_{ord}$ is an isomorphism, or equivalently, whether φ'_{ord} is arithmetic. Hida's theory answers this in the affirmative when $n = 2$, but we know almost nothing in this direction when $n \geq 3$. The eigenpacket φ'_{ord} deserves to be of interest independent of whether or not it is arithmetic. For example, φ'_{ord} parametrizes p-ordinary torsion in the cohomology of the $\mathbb{Z}_p$-lattices L_λ, $\lambda \in \mathcal{W}^+$. If $\varphi'_{ord} \longrightarrow \varphi_{ord}$ is an isomorphism, then it can be shown that every eigenpacket occurring in the p-ordinary torsion must be congruent to some p-ordinary arithmetic $\overline{\mathbb{Q}}_p$-eigenpacket.

Definition 4.4. *We define the p-ordinary eigenscheme to be the formal scheme* $X_{ord} := Spf(R_{ord})$. *We also let* $\mathcal{X}_{ord} := \mathcal{X}_{\varphi_{ord}}$ *and call this the p-ordinary eigenvariety.*

If $\mathcal{O}_K$ is the ring of integers in a finite extension $K/\mathbb{Q}_p$ then any continuous homomorphism $R_{ord} \longrightarrow K$ takes values in $\mathcal{O}_K$. Hence the canonical map

$$X_{ord}(\mathcal{O}_K) \xrightarrow{\sim} \mathcal{X}_{ord}(K)$$

is a bijection. Thus the scheme X_{ord} remembers the K-points of $\mathcal{X}_{ord}$. But the scheme X_{ord} contains additional information related to congruences and this will be important to us below in the definition of quasi-cuspidality. For $x \in X_{ord}(K)$ we let P_x be the kernel in R_{ord} of x. Then P_x is a prime ideal in R_{ord}.

Definition 4.5. *If* $x, x' \in \mathcal{X}_{ord}(\overline{\mathbb{Q}}_p)$ *are two* $\overline{\mathbb{Q}}_p$-points, *then we say* x *and* x' *are* congruent *if* P_x *and* $P_{x'}$ *are contained in the same maximal ideal of* R_{ord}.

We note that two points $x, x' \in \mathcal{X}_{ord}(\overline{\mathbb{Q}}_p)$ are congruent if and only if there is an automorphism τ of $\overline{\mathbb{Q}}_p/\mathbb{Q}_p$ such that for every $\alpha \in \mathcal{H}$ we have the congruence $(\varphi_{ord})_x(\alpha) \equiv \tau\big((\varphi_{ord})_{x'}(\alpha)\big)$ modulo the maximal ideal of the integers of $\overline{\mathbb{Q}}_p$. This is the usual notion of congruence between two systems of Hecke eigenvalues. Indeed, since R_{ord} is complete and semilocal, the set of maximal ideals of R_{ord} is in one-one correspondence with the set of connected components of X_{ord} and these are in one-one correspondence with the set of congruence classes of points on $\mathcal{X}_{ord}$.

We close this section by showing that φ_{ord} is isomorphic to $\mathrm{Tw}(\varphi_{ord})$ and deriving some simple consequences.

Theorem 4.6. *The unique morphism* $\varphi_{ord} \longrightarrow \mathrm{Tw}(\varphi_{ord})$ *is an isomorphism. Moreover, for every specialization* φ_0 *of* φ_{ord} *at a point of* $\mathcal{X}_{ord}(\overline{\mathbb{Q}}_p)$, *there is a morphism*

$$\varphi_{ord} \longrightarrow \widetilde{\mathrm{Tw}}(\varphi_0)$$

where $\widetilde{\mathrm{Tw}}(\varphi_0)$ *is the universal twist of* φ_0 *defined in section 3.*

Proof: Recall that if A is a Λ-module, then $A(\mathbf{1})$ denotes its twist by det (Definition 3.2). Let $\mathbb{D} \cong \mathbb{D}(X)$ be the $\Lambda[S]$-module described in section 2. We define the Λ-morphism $\mathrm{Tw} : \mathbb{D} \longrightarrow \mathbb{D}(\mathbf{1})$ by sending a measure $\mu \in \mathbb{D}$ to the measure $\mathrm{Tw}(\mu) \in \mathbb{D}(\mathbf{1})$ defined by the integration formula

$$\int_X f(x)d\mathrm{Tw}(\mu)(x) = \int_X \det(x)f(x)d\mu(x)$$

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for $f \in C(X)$. Clearly Tw is an isomorphism. On the other hand, for $\sigma \in S$ we have $\mathrm{Tw}(\mu|\sigma) = \det(\sigma)\mathrm{Tw}(\mu)|\sigma$. In particular, Tw is a Γ-morphism and therefore induces a Λ-isomorphism

$$\mathbb{H} \xrightarrow{\sim} \mathbb{H}(\mathbf{1}).$$

Conjugation induces a Λ-isomorphism $\mathrm{Tw} : \mathrm{End}_\Lambda(\mathbb{H}) \xrightarrow{\sim} \mathrm{End}_\Lambda(\mathbb{H}(\mathbf{1}))$. A computation shows that the diagram

$$
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\mathrm{Tw}} & \mathcal{H}(\mathbf{1}) \\
\downarrow & & \downarrow \\
\mathrm{End}_\Lambda(\mathbb{H}) & \xrightarrow{\mathrm{Tw}} & \mathrm{End}_\Lambda(\mathbb{H}(\mathbf{1}))
\end{array}
$$

is commutative where the vertical arrows are both given by φ'_{ord}. Hence Tw induces an isomorphism $\mathrm{Tw} : \varphi'_{ord} \longrightarrow \varphi'_{ord}$. By Proposition 3.3, this morphism preserves the ideal I_{arith} and therefore induces an isomorphism $\mathrm{Tw} : \varphi_{ord} \longrightarrow \mathrm{Tw}(\varphi_{ord})$.

Now let φ_0 be the specialization of φ_{ord} at a point $x_0 \in \mathcal{X}_{ord}(\overline{\mathbb{Q}}_p)$. For each integer n, let $x_n := \mathrm{Tw}^n(x_0) \in \mathcal{X}_{ord}(\overline{\mathbb{Q}}_p)$ and note that we then have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\varphi_{ord}} & R_{ord} \\
\downarrow & & \downarrow \\
\overline{\mathbb{Q}}_p[[\mathbb{Z}_p^\times]] & \xrightarrow{\psi_n^\dagger} & \overline{\mathbb{Q}}_p
\end{array}
$$

where the left vertical arrow is $\widetilde{\mathrm{Tw}}(\varphi_0)$, the right vertical arrow is x_n, and $\psi_n^\dagger$ is the map whose structure character is the nth power character $\mathbb{Z}_p^\times \longrightarrow \overline{\mathbb{Q}}_p^\times$. But since $\bigcap_n \ker(\psi_n^\dagger) = \{0\}$, we see that $\widetilde{\mathrm{Tw}}(\varphi_0)$ maps every $\alpha \in \ker(\varphi_{ord})$ to 0. Finally, since φ_{ord} is surjective, we conclude that $\widetilde{\mathrm{Tw}}(\varphi_0)$ factors through φ_{ord}. This gives us a morphism $\varphi_{ord} \longrightarrow \widetilde{\mathrm{Tw}}(\varphi_0)$. This completes the proof of Theorem 4.6.

If $\varphi = (\varphi_{ord})_x$ is the eigenpacket associated to an arbitrary point $x \in \mathcal{X}_{ord}(\overline{\mathbb{Q}}_p)$, then the last theorem gives us a morphism $R_{ord} \longrightarrow \overline{\mathbb{Q}}_p[[\mathbb{Z}_p^\times]]$. If we compose this morphism with the morphism associated to the canonical projection $\mathbb{Z}_p^\times \longrightarrow 1 + p\mathbb{Z}_p$ (or to $1 + 4\mathbb{Z}_p$ if $p = 2$) we obtain a Λ-morphism

$$\tilde{x} : R_{ord} \longrightarrow \overline{\mathbb{Q}}_p[[1 + p\mathbb{Z}_p]]$$

whose kernel is a prime ideal $P_{\tilde{x}} \subseteq R_{ord}$.

Definition 4.7. For arbitrary $x \in \mathcal{X}_{ord}(\overline{\mathbb{Q}}_p)$ we let $Z(\tilde{x}) \subseteq \mathcal{X}_{ord}$ be the irreducible Zariski closed subset of $\mathcal{X}_{ord}$ cut out by $P_{\tilde{x}}$.

We have the following corollary of Theorem 4.6.

Corollary 4.8. *If Z is an irreducible component of $\mathcal{X}_{ord}$ and $x \in \mathcal{X}_{ord}(\overline{\mathbb{Q}}_p)$ then $x \in Z(\overline{\mathbb{Q}}_p) \Longrightarrow Z(\tilde{x}) \subseteq Z$.*

Proof: Since R_{ord} is finite over Λ, we have $\mathcal{X}_{ord}$ is a noetherian topological space (with the Zariski topology), hence has only finitely many irreducible components.

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Consider the powers Tw^n of the twist morphism $\text{Tw} : \mathcal{X}_{ord} \longrightarrow \mathcal{X}_{ord}$ of Theorem 4.6 for $n \equiv 0$ modulo $(p-1)$ (or modulo 2 if $p = 2$). Clearly, $\text{Tw}^n(Z)$ is an irreducible component of $\mathcal{X}_{ord}$ for every such n, thus there is an integer $N \equiv 0$ modulo $(p-1)$ (again modulo 2 if $p = 2$) such that $\text{Tw}^n(Z) = \text{Tw}^m(Z)$ whenever $n \equiv m$ modulo N. In particular, we have $\text{Tw}^n(Z) = Z$ whenever $N|n$. From this it follows that the set $\{\,\text{Tw}^n(x)\,|\,N|n\,\} \subseteq Z(\overline{\mathbb{Q}}_p)$. But this set is a Zariski dense subset of $Z(\tilde{x})$. Therefore $Z(\tilde{x}) \subseteq Z$ and Corollary 4.8 is proved.

Theorem 4.9. *Let φ_0 be a p-ordinary arithmetic $\overline{\mathbb{Q}}_p$-eigenpacket and let $x_0 \in \mathcal{X}_{ord}^+$ be the corresponding point. Then φ_0 is rigid if and only if $Z(\tilde{x}_0)$ is an irreducible component of $\mathcal{X}_{ord}$.*

Proof. If φ_0 is rigid and Z is an irreducible component of $\mathcal{X}_{ord}$ containing x_0, then by the definition of rigidity, Z must be a twist-deformation of x_0. Hence every arithmetic point of Z is a twist of x_0 and therefore every arithmetic point of Z is in $Z(\tilde{x}_0)$. Since the arithmetic points of Z are Zariski dense in Z, we conclude that $Z \subseteq Z(\tilde{x}_0)$. So by Corollary 4.8, we have $Z = Z(\tilde{x}_0)$ as claimed.

Conversely, suppose $Z(\tilde{x}_0)$ is an irreducible component of $\mathcal{X}_{ord}$ and let (φ, z_0) be any arithmetic deformation of φ_0. Since φ_0 is ordinary, we can choose an irreducible affinoid neighborhood Z of z_0 in $\mathcal{X}_\varphi$ such that φ_Z is p-ordinary. Hence there is a morphism $f : \varphi_{ord} \longrightarrow \varphi_Z$ such that $f^\#(z_0) = x_0$. Since Z is irreducible, we have also $f^\#(Z)$ is irreducible in $\mathcal{X}_{ord}$. Let W be an irreducible component containing $f^\#(Z)$. By Corollary 4.8, $Z(\tilde{x}_0) \subset W$. Since $Z(\tilde{x}_0)$ is an irreducible component, $Z(\tilde{x}_0) = W$. Thus $f^\#(Z) \subseteq Z(\tilde{x}_0)$. Hence φ_Z is a twist of φ_0 and it follows that (φ, z_0) is a twist deformation of φ_0. This proves φ_0 is rigid and Theorem 4.9 is proved.

SECTION 5: PROOF OF LEMMA 4.3.

In this section, we prove a general cohomological result (Theorem 5.1) and deduce Lemma 4.3 as a consequence.

Let M be a profinite $\Lambda[S]$-module and let $H^*(M)^0$ be the ordinary part of the cohomology $H^*(\Gamma, M)$ with respect to the operator $T(\pi)$. Then we may (and do) regard $H^*(M)^0$ as an $\mathcal{H}$-module. Define $R(M)$ to be the image of $\mathcal{H}$ in $\text{End}_\Lambda(H^*(M)^0)$ and define $A(M)$ to be $R(M)^{red}$, i.e. $R(M)$ modulo its nilradical.

Theorem 5.1. *Let M be a profinite $\Lambda[S_0]$-module and suppose $H^*(M)^0$ is a finitely generated Λ-module. Let I be an ideal in Λ that is generated by a finite M-regular sequence. Then $H^*(M/IM)^0$ is a finitely generated Λ/I-module. Moreover there is a natural isomorphism*

$$R(M)/Rad_{R(M)}(I) \cong A(M/I)$$

where $Rad_{R(M)}(I)$ is the radical of $IR(M)$ in $R(M)$.

Before proving Theorem 5.1, we first indicate how it implies Lemma 4.3.

Proof of Lemma 4.3. If we take $M = \mathbb{D}$ then we have $R(\mathbb{D}) = R'_{ord}$. Moreover, we know (see Lemma 2.2) that for each dominant integral weight λ, the ideal $I_\lambda \subseteq \Lambda$ is generated by a $\mathbb{D}$-regular sequence and that $\mathbb{D}/I_\lambda\mathbb{D} \cong \mathbb{D}_\lambda$. Hence, by

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Theorem 5.1, we have $A(\mathbb{D}_\lambda) \cong R'_{ord}/Rad_{R'_{ord}}(I_\lambda)$. Moreover, from Lemma 2.3 we have $A(L_\lambda) \cong A(\mathbb{D}_\lambda)$. Thus we have

$$A(L_\lambda) \cong R'_{ord}/Rad_{R'_{ord}}(I_\lambda)$$

from which Lemma 4.3 is an immediate consequence.

We now turn to the proof of Theorem 5.1.

Proof of Theorem 5.1. Our proof is by induction on the length of an M-regular sequence of generators for I. We begin with the case where I is principal, generated by an M-regular element $\alpha \in \Lambda$, i.e. an element α such that M has no α-torsion. In this case, we have an exact sequence

$$0 \longrightarrow M \overset{\alpha}{\longrightarrow} M \longrightarrow M/\alpha M \longrightarrow 0.$$

Now pass to the long exact cohomology sequence, we obtain an exact sequence

$$0 \longrightarrow H/\alpha H \longrightarrow H^*(M/\alpha M)^0 \longrightarrow H[\alpha] \longrightarrow 0$$

where $H := H^*(M)^0$ and $H[\alpha]$ is the α-torsion in H. We first construct a homomorphism $R(M) \longrightarrow A(M/\alpha M)$. For this, we let $x \in \mathcal{H}$ be such that x annihilates H. Then the last exact sequence implies x^2 annihilates $H^*(M/\alpha M)^0$. Hence x maps to an element of the nilradical of $R(M/\alpha M)$ and it follows that x maps to 0 in $A(M/\alpha M)$. Thus the canonical map $\mathcal{H} \longrightarrow A(M/\alpha M)$ factors through the canonical map $\mathcal{H} \longrightarrow R(M)$. Hence we have a canonical surjective map

$$\tilde{\varphi} : R(M) \longrightarrow A(M/\alpha M).$$

We need to show that $\ker(\tilde{\varphi}) = Rad_{R(M)}(\alpha)$. The inclusion $\supseteq$ is obvious.

So let $x \in \ker(\tilde{\varphi})$. From the above exact sequence, we conclude that $xH \subseteq \alpha H$. Since, by hypothesis, H is finitely generated over Λ, there is a positive integer m such that $H[\alpha^{m+1}] = H[\alpha^m]$. In particular, we see that $\alpha^m H$ has no α-torsion and from this it follows that $\alpha : \alpha^m H \longrightarrow \alpha^{m+1} H$ is an isomorphism of Λ-modules. Let $\beta : \alpha^{m+1} H \longrightarrow \alpha^m H$ be the inverse map. Let $y \in \text{End}_\Lambda(H)$ be the composition

$$y : H \overset{x^{m+1}}{\longrightarrow} \alpha^{m+1} H \overset{\beta}{\longrightarrow} \alpha^m H \subseteq H.$$

Then $\alpha y = x^{m+1}$.

Now define

$$R' := \{\, y \in \text{End}_\Lambda(H) \,|\, \exists k \in \mathbb{Z}^+, z \in R(M) \text{ such that } \alpha^k y = \alpha^m z \,\}.$$

The endomorphism y constructed in the last paragraph is an element of R'. Clearly, R' is a finite Λ-algebra containing $R(M)$. Moreover, for every element of $\rho \in R'$ we have $\alpha^k \rho \in R(M)$ for some $k \geq 0$. Since R' is finitely generated over Λ, there is an exponent N such that $\alpha^N R' \subseteq R(M)$.

Now consider $x^{(m+1)(N+1)} = (\alpha y)^{N+1} = \alpha(\alpha^N y^{N+1})$. Since $y^{N+1} \in R'$ we have $z := \alpha^N y^{N+1} \in R(M)$. Hence $x^{(m+1)(N+1)} \in \alpha R(M)$. This proves $x \in Rad_{R(M)}(\alpha)$. Hence $\ker(\varphi) \subseteq Rad_{R(M)}(\alpha)$. This completes the proof in the special case where I is generated by a single M-regular element of Λ.

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Now we suppose $r \geq 1$ and that the theorem is true whenever I is generated by an M-regular sequence of length r. Let $\alpha_1, \ldots, \alpha_r, \alpha$ be an M-regular sequence of length $r+1$. Let I be the ideal generated by $\alpha_1, \ldots, \alpha_r$ in Λ and let J be the ideal generated by I and α. We define ψ to be the composition of the surjective homomorphisms

$$\psi : R(M) \xrightarrow{\varphi} R(M/IM) \longrightarrow A(M/JM).$$

Let $x \in \ker(\psi)$. Then by the principal case proved in the last paragraph, we have $y := \varphi(x) \in Rad_{R(M/IM)}(\alpha)$. Thus there is an $m \in \mathbb{Z}^+$ such that $y^m \in \alpha R(M/IM)$. This means $\varphi(x^m) \in \alpha R(M/IM)$. So there is an element $z \in R(M)$ such that $\varphi(x^m) = \alpha\varphi(z)$. It then follows from the induction hypothesis that $x^m - \alpha z \in Rad_{R(M)}(I)$. There is therefore a positive integer N such that

$$(x^m - \alpha z)^N \in IR(M).$$

From this we see at once that $x^{mN} \in JR(M)$. Hence $x \in Rad_{R(M)}(J)$. This proves $\ker(\psi) \subseteq Rad_{R(M)}(J)$. But the opposite inclusion is immediate. Hence $\ker(\psi) = Rad_{R(M)}(J)$. This completes the proof of Theorem 5.1.

Section 6: Cuspidal and Quasi-cuspidal Eigenpackets

In this section we recall briefly the connection between cohomology and automorphic representations and review some results from the theory of mod p cohomology of congruence subgroups Γ of $SL(n, \mathbb{Z})$ in order to have a sufficient condition for arithmetic eigenpackets to be cuspidal automorphic.

We denote the adeles of $\mathbb{Q}$ by $\mathbb{A}$. Let $\Gamma = \Gamma_0(N)$ for some N, M a compact open subgroup of $GL(n, \mathbb{A}_f)$ whose intersection with $GL(n, \mathbb{Q})$ is Γ. Let $\tilde{\mathfrak{g}}$ denote the Lie algebra of $SL(n, \mathbb{R})$, $K = O(n)$, Z the center of $SL(n, \mathbb{R})$. Let E be any finite dimensional irreducible complex representation of $GL(n, \mathbb{R})$.

The cuspidal cohomology $H^*_{cusp}(\Gamma, E)$ is defined in [Borel 83]. By Lemma 3.15, page 121 of [Clozel 90], there is a canonical surjective map

$$\Psi : \bigoplus_{\Pi} H^*(\tilde{\mathfrak{g}}, K; \Pi_\infty \otimes E) \otimes \Pi_f^M \to H^*_{cusp}(\Gamma, E)$$

where Π runs over a certain finite set of cuspidal automorphic representations of $GL(n, \mathbb{A})$.

If $\alpha \in H^*_{cusp}(\Gamma, E)$ is an $\mathcal{H}$-eigenclass, then by the strong multiplicity one theorem for $GL(n)$, there is a unique Π_α such that $\alpha \in \Psi(H^*(\tilde{\mathfrak{g}}, K; \Pi_{\alpha,\infty} \otimes E) \otimes \Pi_{\alpha,f}^M)$.

We fix an isomorphism of $\mathbb{C}_p$ and $\mathbb{C}$. Recall that Γ_0 denotes the subset of Γ consisting of those matrices that are upper triangular modulo p. If $\alpha \in H^*(\Gamma_0, L_\lambda)$, we say that α is cuspidal if its image in $H^*(\Gamma_0, L_\lambda \otimes \mathbb{C}) \simeq H^*(\Gamma_0, L_\lambda) \otimes_{\mathbb{Z}_p} \mathbb{C}$ is cuspidal.

Definition 6.0. *An $\mathcal{O}_K$-valued eigenpacket is said to be cuspidal if it occurs as the system of $\mathcal{H}$-eignevalues on a cuspidal cohomology class. A point on $\mathcal{X}_{ord}(K)$ is cuspidal if φ_x is cuspidal.*

We will need the following fact: set $n = 3$ and let ∂Y denote the boundary of the Borel-Serre compactification of $\Gamma \backslash GL(n, \mathbb{R})/KZ$. Then

$$H^3_{cusp}(\Gamma, E) \oplus H^3(\partial Y, E) \simeq H^3(\Gamma, E).$$

See for instance Proposition 2.1 and its proof in [Ash-Grayson-Green 84] which, though stated there for trivial coefficients, remains valid for E-coefficients.

If φ is an $\mathcal{O}_K$-valued eigenpacket such that $\varphi(T(\ell,k)) = a(\ell,k)$ then the Hecke polynomial of φ at ℓ is defined to be

$$P_{\varphi,\ell}(X) = \sum_{k=0}^{n} (-1)^k a(\ell,k) \ell^{k(k-1)/2} X^k \in \mathcal{O}_K[X].$$

If $x \in \mathcal{X}_{ord}(K)$ then we will write $P_{x,\ell}$ for $P_{\varphi_x,\ell}$.

Definition 6.1. *An $\mathcal{O}_K$-valued eigenpacket φ is said to be ℓ-quasicuspidal if $P_{\varphi,\ell}$ is irreducible modulo the maximal ideal of $\mathcal{O}_K$. We say φ is quasicuspidal if φ is ℓ-quasicuspidal for some prime $\ell \neq p$.*

We say a point $x \in \mathcal{X}_{ord}(K)$ is ℓ-quasicuspidal (resp. quasicuspidal) if φ_x is ℓ-quasicuspidal (resp. quasicuspidal).

The following proposition follows immediately from the definitions.

Proposition 6.2. *If $x \in \mathcal{X}_{ord}(K)$ is quasicuspidal then every point in the connected component of x in $\mathcal{X}_{ord}(K)$ is also quasicuspidal.*

Conjecture 6.3. *Every quasicuspidal arithmetic eigenpacket is cuspidal.*

Theorem 6.4. *If $n = 2$ or 3 then conjecture 6.3 is true.*

Proof. This follows from Proposition 3.2.1 of [Ash-Stevens 86] which says that a Hecke eigenclass in $H^3(\partial Y, E)$ has all its Hecke polynomials reducible plus the fact that a non-cuspidal Hecke eigenclass must restrict nontrivially to the Borel-Serre boundary, if $n \leq 3$.

SECTION 7: REMOVING p FROM THE LEVEL

In this section only, in order to conform with the convention of the references, all representation spaces will be left modules.

Let I denote the Iwahori subgroup of $GL(n, \mathbb{Z}_p)$ consisting of all matrices which are upper triangular modulo p. When $n = 3$, our computers are not powerful enough to compute the cohomology of $\Gamma_0 := \Gamma \cap I$. We can only compute for the larger group Γ. Therefore we need a theorem that allows us to remove p from the level of a cohomology class.

Let $G = GL(n)$ and let (B, T) be the pair of upper triangular and diagonal matrices respectively. We use the corresponding ordering on the roots, so that the positive roots are those occurring in B. Let δ be the character on T which is the p-adic absolute value of the product of the positive root characters: $\delta(\text{diag}(t_1, \ldots, t_n)) = |t_1|^{n-1} |t_2|^{n-3} \cdots |t_n|^{1-n}$.

Recall that we have identified the Hecke algebras $\mathcal{H}$ and $\mathcal{H}_0$ for Γ and Γ_0, and so $\mathcal{H}$ acts on the cohomology of both Γ and Γ_0. We then have

Theorem 7.1. *Let Γ be a congruence subgroup of $GL(n, \mathbb{Z})$ of level prime to p and let $\Gamma_0 = \Gamma \cap I$ as above. Let λ be the highest weight $(a_1, a_2, \ldots, a_n)$, so that $a_1 \geq a_2 \geq \cdots \geq a_n$. Let ϕ be a packet of Hecke eigenvalues (for $\mathcal{H}$) that occurs in $H^*_{cusp}(\Gamma_0, L_\lambda(\mathbb{C}))^0$. Suppose $a_1 > a_2 > \cdots > a_n$. Then there is an $\mathcal{H}$-eigenclass*

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$\alpha \in H^*_{cusp}(\Gamma, L_\lambda(\mathbb{C}))$ *such that for any $l \neq p$ and any $i = 1, \ldots, n$, $T(l,i)\alpha = \phi(T(l,i))\alpha$.*

Proof. Let $|\ |$ denote the p-adic absolute value normalized so that $|p| = p^{-1}$. Let $\alpha \in H^*_{cusp}(\Gamma_0, L_\lambda)^0$ be an ordinary cuspidal cohomology class which is a Hecke eigenclass with eigenvalues given by ϕ. Let Π be the irreducible automorphic representation of $GL(n, \mathbb{A})$ associated to α and write $\Pi_f = \otimes \Pi_v$.

Then Π_p has a nonzero I-invariant vector and therefore is isomorphic to a subrepresentation of the induced representation $\mathbb{I}(\chi)$ from $B(\mathbb{Q}_p)$ to $G(\mathbb{Q}_p)$ of some unramified character χ on $T(\mathbb{Q}_p)$ (Theorem 3.8 in [Cartier 79]). This is normalized induction, so that $\mathbb{I}(\chi)$ consists of the locally constant functions $f : G(\mathbb{Q}_p) \to \mathbb{C}$ such that $f(bg) = \delta(b)^{1/2}\chi(b)f(g)$ for $b \in B$ and $g \in G$ (where χ is extended from T to B by making it trivial on the unipotent radical of B.)

By a straightforward but rather complicated explicit computation in the induced representation, we can determine the unramified character χ given the eigenvalues of the Hecke operators at p. Let $U(p,m)$ denote the Hecke operator $U(1, \ldots, 1, p, \ldots, p)$ with m copies of p, $m = 0, \ldots, n$. We may choose α so that it is also an eigenclass for all the $U(p,m)$ operators. Let β_m be the eigenvalue of $U(p,m)$ on the cohomology class α with respect to the $\star$-action. By assumption, the algebraic integers β_i are p-adic units. Note that $\beta_0 = \beta_n = 1$, since $U(p,0)$ and $U(p,n)$ act via the $\star$-action as the identity.

To compare the cohomology with the automorphic theory, we have to remove the $\star$-action. With respect to the usual action, the eigenvalue of $U(p,m)$ on α is $p^{a_n + a_{n-1} + \cdots + a_{n-m+1}}\beta_m$.

Let d_m be the diagonal matrix with 1's along the diagonal except for a p at the m-th place. The explicit computation alluded to above gives the following values, for $m = 1, \ldots, n$:
$$\chi\delta^{1/2}(d_m) = p^{a_{n-m+1} - n + 2m - 1}\beta_m\beta_{m-1}^{-1}.$$

It follows that if $r_m = a_{n-m+1} - (n - 2m + 1)/2$, then

$$\chi(d_m) = p^{r_m}\beta_m\beta_{m-1}^{-1}.$$

Define characters χ_i of $\mathbb{Q}_p$ by $\chi(\mathrm{diag}\,(t_1, \ldots, t_n)) = \chi_1(t_1) \ldots \chi_n(t_n)$. Since χ is unramified, each $\chi_i(t) = |t|^{c_i}$ for some complex number c_i.

It is a result of Bernstein and Zelevinsky ([Bernstein-Zelevinsky 77], theorem 4.2) that $\mathbb{I}(\chi)$ is irreducible unless $\chi_i\chi_j^{-1} = |\ |$ for some $i \neq j$. See section 1 of [Kudla 94] for a discussion of question of reducibility of represenations induced from parabolic subgroups.

We compute
$$\chi_i\chi_j^{-1}(p) = p^{r_i - r_j}\beta_i\beta_{i-1}^{-1}\beta_j^{-1}\beta_{j-1}.$$

Note that since $a_1 > a_2 > \cdots > a_n$, the r_m are a strictly increasing sequence of integers with gaps of at least 2 between successive terms. Taking into account the fact that the β_m are p-adic units, we see that $\chi_i\chi_j^{-1}(p)$ never equals $p^{-1} = |p|$. So we conclude that $\mathbb{I}(\chi)$ is irreducible and so $\Pi_p = \mathbb{I}(\chi)$.

Let Ψ be the map from $(\tilde{\mathfrak{g}}, K)$ cohomology discussed in section 6. As in that section, M denotes a compact subgroup of $G(\mathbb{A}_f)$ whose intersection with $G(\mathbb{Q})$ is Γ. We let $M_0 = \{y \in M \mid y_p \in I\}$. Then $\alpha = \Psi(c \otimes x)$ for some generator c of $H^*(\tilde{\mathfrak{g}}, K; \Pi_{\alpha,\infty} \otimes E)$ and for some $x = \otimes x_v \in \Pi_f^{M_0}$.

It follows from (c) p. 152 of [Cartier 79] that Π_p contains a vector x'_p fixed under $GL(n, \mathbb{Z}_p)$. (So in fact Π is unramified at p.) Set $x' = \otimes x'_v$ where $x'_v = x_v$ if $v \neq p$. Then $\alpha' = \Psi(c \otimes x')$ is a Hecke eigenclass in $H^*_{cusp}(\Gamma, L_\lambda(\mathbb{C}))$ with the same packet of $\mathcal{H}'$-eigenvalues as α.

Section 8: Essential Self-duality

We begin this section with some considerations of selfduality. If B is a ring and π a representation of $GL(n, B)$, the contragredient of π is the representation π' defined by $\pi'(g) = \pi(^tg^{-1})$. We say that π is "essentially selfdual" if there is a character χ of $B^\times$ such that π' is isomorphic to $\pi \otimes (\chi \circ \det)$.

In parallel to this, if H is a Hecke algebra over B with involution ι and $\phi : H \to A$ is a B-homorphism, we will say that ϕ is "essentially selfdual" if there is a character χ of $A^\times$ such that for every double coset T in H, $\phi(\iota(T)) = \chi(\det(T))\phi(T)$.

Definition 8.1. a) If (Γ, S) is a Hecke pair for $GL(n)$ and E is a finite dimensional S-module over a ring of characteristic 0, a Hecke eigenclass in $H^*(\Gamma, E)$ is called an essentially selfdual cohomology class if its associated automorphic representation (as in §6) is an essentially selfdual representation.

b) If ϕ is an $\mathcal{H}$-eigenpacket (as in §3), it is called an essentially selfdual eigenpacket if $\phi|\mathcal{H}'$ is essentially selfdual, where $\mathcal{H}'$ is the Hecke algebra of the semigroup S' defined in §1 and ι is the involution of it given by transpose-inverse composed with conjugation by the matrix $\mathrm{diag}(N, 1, \ldots, 1)$.

It follows from the definitions that an essentially selfdual cohomology eigenclass affords an essentially selfdual eigenpacket.

By theorem 6.12 II in [Borel-Wallach 00], the cuspidal cohomology of an arithmetic group with coefficients in a finite dimensional irreducible rational module E over a field L of characteristic 0 vanishes unless E is essentially selfdual as a L-module. We shall call such E an essentially selfdual coefficient module and its highest weight an essentially selfdual weight.

It's easy to see that an essentially selfdual eigenpacket has an essentially selfdual weight. The converse is not true, as shown by the examples computed in [Ash-Grayson-Green 84].

The (contravariant) dominant integral weights with respect to (B^{opp}, T) are given by n-tuples of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Such an n-tuple corresponds to the character $t := \mathrm{diag}(t_1, t_2, \ldots, t_n) \mapsto t^\lambda := t_1^{\lambda_1} t_2^{\lambda_2} \cdots t_n^{\lambda_n}$ on the torus T. The highest weight of the contragredient of V_λ is the dominant integral weight $\lambda^\vee := (-\lambda_n, \ldots, -\lambda_1)$. So λ is essentially self-dual if and only if $\lambda_i + \lambda_{n+1-i} = \lambda_j + \lambda_{n+1-j}$ whenever $1 \leq i, j \leq n$.

Definition 8.2. We say that a point $k = (k_1, k_2, \ldots, k_n) \in \mathcal{W}(K)$ of the weight space is essentially self dual if $k_i + k_{n+1-i} = k_j + k_{n+1-j}$ whenever $1 \leq i, j \leq n$. We let $\mathcal{W}^{esd}$ be the set of all essentially self-dual weights in $\mathcal{W}$. We also define

$$\mathcal{X}^{esd}_{ord} := \{ x \in \mathcal{X}_{ord} \mid \mathrm{wt}(x) \in \mathcal{W}^{esd} \}.$$

We note that $\mathcal{W}^{esd}$ is a closed subspace of $\mathcal{W}$ of dimension $[n/2] + 1$ and that $\mathcal{X}^{esd}_{ord}$ is therefore a closed subspace of $\mathcal{X}_{ord}$ of dimension $\leq [n/2] + 1$.

Theorem 8.3. *Every cuspidal point of $\mathcal{X}_{ord}$ is contained in $\mathcal{X}_{ord}^{esd}$.*

Proof. This is immediate from the Borel-Wallach theorem and the fact that $\mathcal{X}_{ord}^{esd}$ is Zariski closed.

We now suppose $n = 3$. We will use the notation $\lambda(g) = (2g, g, 0)$ for $g \geq 0$. Any essentially selfdual highest weight is of the form $\lambda(g) \otimes \det^j$ for some $g \geq 0$ and $j \in \mathbb{Z}$.

Theorem 8.4. *Let $g_0 \geq 0$ and set $\lambda_0 = \lambda(g_0)$. Let $x_0 \in \mathcal{X}_{ord}(\overline{\mathbb{Q}}_p)$ be a quasi-cuspidal arithmetic eigenpacket of weight λ_0 and suppose x_0 is not arithmetically rigid. Then for every integer $g \equiv g_0$ modulo $p - 1$ (or modulo 2 if $p = 2$), there is a cuspidal automorphic point $x \in \mathcal{X}_{ord}(\overline{\mathbb{Q}}_p)$ of weight $\lambda(g)$ that is congruent to x_0.*

Proof. Since x_0 is not arithmetically rigid and let Z be an irreducible component of $\mathcal{X}_{ord}$ containing x_0. Then Corollary 4.8 and Theorem 4.9 tell us that $Z(\tilde{x}_0)$ is properly contained in Z. In particular, we know that $\dim Z \geq 2$. On the other hand, since x_0 is quasi-cuspidal, every arithmetic point on Z is quasi-cuspidal and is therefore cuspidal (theorem 6.4). So theorem 8.3 tells us $Z \subseteq \mathcal{X}_{ord}^{esd}$. But since $n = 3$, we know $\mathcal{X}_{ord}^{esd}$ has dimension ≤ 2. It follows that Z is an irreducible component of $\mathcal{X}_{ord}^{esd}$ and that $\dim Z = 2$. Since R_{ord} is finite over Λ, the image of Z under the weight map is a Zariski-closed irreducible subset of $\mathcal{W}^{esd}$ also having dimension 2. But $\mathcal{W}^{esd}$ is 2-dimensional and therefore the image of Z is precisely the connected component of $\mathcal{W}$ containing λ_0.

But if g is a non-negative integer $g \equiv g_0$ modulo $p - 1$ (or modulo 2 if $p = 2$), then $\lambda(g)$ is in the same connected component of $\mathcal{W}$ as λ_0. Thus, there is a point $x \in Z(\overline{\mathbb{Q}}_p)$ lying over $\lambda(g)$. By lemma 4.3, x is arithmetic. Clearly x and x_0 are congruent (see definition 4.5 and the accompanying remarks). Hence x is quasi-cuspidal and is therefore cuspidal automorphic of weight $\lambda(g)$. This completes the proof of the theorem.

The following corollary is immediate from the theorem and the definitions. Recall from the introduction that we denote by V_g the representation $V_{\lambda(g)}$ of highest weight $\lambda(g)$.

Corollary 8.5. *Let φ_0 be a quasi-cuspidal arithmetic eigenpacket of weight $\lambda(g_0)$ occurring in $H^3(\Gamma, V_{g_0}(K))$ and suppose φ_0 is not arithmetically rigid. Then for every integer $g \equiv g_0$ modulo $p - 1$ (or modulo 2 if $p = 2$), there is a finite extension L/K and a cuspidal automorphic eigenpacket φ occurring in $H^3(\Gamma, V_g(L))$ such that φ is congruent to φ_0.*

SECTION 9: EXAMPLES OF p-ADICALLY RIGID CLASSES VIA COMPUTATION

In this section, we assume $n = 3$. According to Corollary 8.5, to check that a given quasi-cuspidal arithmetic eigenpacket is arithmetically rigid, it suffices to show $H^3_{cusp}(\Gamma, V_h(\mathbb{C})) = 0$ for suitable h.

For any cohomology group $H^*()$ with coefficients in a module which is a vector space over a field k, let $h^*()$ denote its dimension over k. Let $s_w(N)$ denote the dimension of the space of classical holomorphic cuspforms of weight w, level N and trivial nebentype for $GL(2)/\mathbb{Q}$.

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Lemma 9.1. *Let q be prime. Then if h is even,*

$$h^3(\Gamma_0(q), V_h(\mathbb{C})) = h^3_{cusp}(\Gamma_0(q), V_h(\mathbb{C})) + 2s_{h+2}(q) + 3$$

and if h is odd,

$$h^3(\Gamma_0(q), V_h(\mathbb{C})) = h^3_{cusp}(\Gamma_0(q), V_h(\mathbb{C})).$$

Proof. This follows from the fact that $h^3_{cusp}(\Gamma, E) + h^3(\partial Y, E) = h^3(\Gamma, E)$ (see section 4). To compute the cohomology of the Borel-Serre boundary, take $\Gamma_0(q)$-invariants in Theorem 7.6 (4) p. 116 in [Schwermer 83] (where $m = q$). Compare Proposition 2.3 and Lemma 3.9 of [Ash-Grayson-Green 84].

Rather than compute over $\mathbb{C}$, we compute over $\mathbb{Z}/r$ for some prime r. In fact we took $r = 149, 151$ or 163. The following lemma follows immediately from the Universal Coefficient Theorem:

Lemma 9.2. *Let $\Gamma = \Gamma_0(q)$ for some odd prime q, and let r be a prime. Then $h^3(\Gamma, V_h(\mathbb{Z}/r)) = h^3(\Gamma, V_h(\mathbb{C})) + e$ for some non-negative integer e.*

Lemma 9.3. *Let $\Gamma = \Gamma_0(q)$ for some odd prime q, $h \geq 0$ be an even integer, and r be a prime. If $h^3(\Gamma, V_h(\mathbb{Z}/r)) = 2s_{h+2}(q) + 3$ then $h^3_{cusp}(\Gamma, V_h(\mathbb{C})) = 0$ and $e = 0$.*

Proof. This follows from the previous two lemmas.

From [Ash-Grayson-Green 84] we derive the following two examples of p-adically rigid cohomology classes. (On the other hand, we have no glimmer of any non-rigid non-essentially-selfdual cohomology classes, consistent with the conjecture we made in the introduction.)

Theorem 9.4.

(1) *Let φ be either of the two eigenpackets in $H^3_{cusp}(\Gamma_0(89), \mathbb{C})$. Then φ is 3-adically arithmetically rigid.*

(2) *Let ψ be either of the eigenpackets in $H^3_{cusp}(\Gamma_0(61), \mathbb{C})$. Then ψ is 5-adically arithmetically rigid up to twist.*

Proof. (1) A simple calculation from the Hecke data given in [Ash-Grayson-Green 84] shows that for $p = 3$, φ is 2-quasicuspidal and ordinary. Therefore by Corollary 8.5, if φ were not 3-adically arithmetically rigid, there would be nontrivial cuspidal cohomology in $H^3_{cusp}(\Gamma_0(89), V_2(\mathbb{C}))$. But the computations reported upon in the introduction to this paper say that $H^3_{cusp}(\Gamma_0(89), V_2(\mathbb{C})) = 0$.

(2) Similarly, for $p = 5$, ψ is 2-quasicuspidal and ordinary. Since our computations say that $H^3_{cusp}(\Gamma_0(61), V_4(\mathbb{C})) = 0$, the same reasoning shows that ψ is 5-adically arithmetically rigid up to twist.

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